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Orthogonal Polynomials on Arcs of the Unit Circle, I

Franz Peherstorfer and Robert Steinbauer*

Johannes Kepler Universität Linz, Institut für Mathematik, A-4040 Linz, Austria
E-mail: franz.peherstorfer@jk.uni-linz.ac.at, robert steinbauer@jk.uni-linz.ac.at.

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Let $E_l = \bigcup_{j=1}^{l} [\varphi_{2j-1}, \varphi_{2j}] \subseteq [0, 2\pi]$, $\mathscr{R}(\varphi) = \prod_{j=1}^{2l} \sin((\varphi - \varphi_j)/2)$ and $1/r(\varphi) = (-1)^j / \sqrt{|\mathscr{R}(\varphi)|}$ for $\varphi \in (\varphi_{2j-1}, \varphi_{2j})$. Furthermore let $\mathscr{V}$, $\mathscr{W}$ be arbitrary real trigonometric polynomials such that $\mathscr{R} = \mathscr{V}\mathscr{W}$ and let $\mathscr{A}(\varphi)$ be a real trigonometric polynomial which has no zero in E_l. First we derive an explicit representation of the Caratheodory function associated with $f(\varphi; \mathscr{W}) = \mathscr{W}(\varphi)/\mathscr{A}(\varphi)\, r(\varphi)$ on E_l. With the help of this result the polynomials $P_n(z)$, which are orthogonal on the set of arcs $\Gamma_{E_l} := \{e^{i\varphi} : \varphi \in E_l\}$ with respect to $f(\varphi; \mathscr{W})$, are completely characterized by a quadratic equation. (In fact a more general case including Dirac-mass points is considered.) This characterization is the basis of all of our further investigations on polynomials orthogonal on several arcs as the description of that measures which generate orthogonal polynomials with periodic or asymptotically periodic reflection coefficients, the explicit representation of the orthogonality measure of the associated polynomials, the asymptotic representation of polynomials orthogonal on Γ_{E_l}, etc. © 1996 Academic Press, Inc.

1. Introduction and Notation

For $n \in \mathbb{N}_0 := \{0, 1, 2, ...\}$ let $\mathbb{P}_n^{\mathbb{C}}$ denote the space of complex algebraic polynomials of degree (abbreviated by ∂) $\leqslant n$, $\mathbb{P}_{-1}^{\mathbb{C}}$ is the set which only contains the zero-polynomial and

$$\Pi_{n/2} := \left\{ \sum_{k=0}^{\lfloor n/2 \rfloor} a_k \cos\left(\frac{n-2k}{2}\varphi\right) + b_k \sin\left(\frac{n-2k}{2}\varphi\right) : a_k, b_k \in \mathbb{R} \right\}$$

denotes the space of real trigonometric polynomials of (integer or half integer) degree. We say $\mathscr{D} \in \Pi_{n/2}$ is of exact degree $\partial\mathscr{D} = n/2$, if $|a_0| + |b_0| \neq 0$. As usual let $\mathbb{P}^{\mathbb{C}}$ and Π denote the set of all complex algebraic and real trigonometric polynomials, respectively.

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Each element $\mathcal{D}$ from $\Pi_{n/2}$ has a representation of the form

$$\mathcal{D}(\varphi)=c\prod_{j=1}^{n}\sin\left(\frac{\varphi-\psi_j}{2}\right),\qquad c\in\mathbb{R},\tag{1.1}$$

where $\psi_j\in[a,a+2\pi)$, $a\in\mathbb{R}$, or the ψ_j's appear in pairs of complex conjugate numbers with real part in $[a,a+2\pi)$. Hence, a real trigonometric polynomial $\mathcal{D}\in\Pi$ which has an even (odd) number of zeros in $[a,a+2\pi)$ has integer (non-integer) degree. $\mathcal{D}\in\Pi$ is called monic if $|c|=1$ in (1.1).

Now let $2l$, $l\in\mathbb{N}$, points $\varphi_1\leqslant\cdots\leqslant\varphi_{2l}$ with $\varphi_{2l}-\varphi_1\leqslant 2\pi$ be given, let

$$E_l:=\bigcup_{j=1}^{l}[\varphi_{2j-1},\varphi_{2j}],\qquad Int(E_l):=\bigcup_{j=1}^{l}(\varphi_{2j-1},\varphi_{2j}),$$

and (1.2)

$$\Gamma_{E_l}:=\{e^{i\varphi}\colon \varphi\in E_l\}$$

and let $\mathcal{R}\in\Pi$ be the monic real trigonometric polynomial which vanishes at the φ_j's (counted according to their multiplicity), i.e.,

$$\mathcal{R}(\varphi)=\prod_{j=1}^{2l}\sin\left(\frac{\varphi-\varphi_j}{2}\right)\qquad\text{with}\quad \mathcal{R}(\varphi)<0\qquad\text{on } Int(E_l)\tag{1.3}$$

(note that φ_j is a zero of multiplicity k, if $\varphi_j=\cdots=\varphi_{j+k-1}$).

Furthermore let $\mathcal{V}\in\Pi_l$, $\mathcal{W}\in\Pi_l$ be a splitting of $\mathcal{R}$, i.e.,

$$\mathcal{R}(\varphi)=\mathcal{V}(\varphi)\,\mathcal{W}(\varphi),\tag{1.4}$$

and in particular each zero of $\mathcal{V}$ and $\mathcal{W}$ is a zero of $\mathcal{R}$. Finally, let $\mathcal{A}\in\Pi$ be an arbitrary real trigonometric polynomial which has no zeros in E_l, i.e.,

$$\mathcal{A}(\varphi)\neq 0\qquad\text{for}\quad \varphi\in E_l.\tag{1.5}$$

This means that $\mathcal{A}$ can be represented in the form

$$\mathcal{A}(\varphi):=c_{\mathcal{A}}\prod_{j=1}^{m^*}\left(\sin\frac{\varphi-\xi_j}{2}\right)^{m_j}\in\Pi,\tag{1.6}$$

where m^*, $m_j\in\mathbb{N}$ and where the ξ_j's are distinct and lie in $\mathbb{C}\backslash E_l$.

In this paper we investigate polynomials orthogonal with respect to weight functions of the form

$$f(\varphi;\mathcal{A},\mathcal{W}):=\begin{cases}\dfrac{\mathcal{W}(\varphi)}{\mathcal{A}(\varphi)\,r(\varphi)}, & \varphi\in E_l\\ 0, & \varphi\notin E_l\end{cases}\tag{1.7}$$

where

$$\frac{1}{r(\varphi)} := \frac{(-1)^j}{\sqrt{|\mathcal{R}(\varphi)|}}, \qquad \varphi \in (\varphi_{2j-1}, \varphi_{2j}), \qquad j = 1, ..., l. \tag{1.8}$$

Let us mention that

$$\operatorname{sgn} \frac{1}{r(\varphi)} = (-1)^l \operatorname{sgn} \prod_{j=1}^{l-1} \sin\left(\frac{\varphi - \varphi_{2j+1}}{2}\right) \qquad \text{for} \quad \varphi \in Int(E_l).$$

Note that $f(\varphi; \mathcal{A}, \mathcal{W})$ is a function which is not necessarily positive on E_l and which has square root singularities at (some of) the endpoints of E_l.

As usual, an algebraic polynomial P_n of degree n is said to be orthogonal with respect to the weight function $f(\varphi; \mathcal{A}, \mathcal{W})$, if

$$\int_{E_l} e^{-ij\varphi} P_n(e^{i\varphi})\, f(\varphi; \mathcal{A}, \mathcal{W})\, d\varphi = 0, \qquad j = 0, ..., n-1. \tag{1.9}$$

In fact we even study a more general class of orthogonality measures which also includes Dirac-mass points (compare (1.17) below). Because of the orthogonality property (1.9) we also speak of orthogonal polynomials on several arcs of the unit circle.

To guarantee that $f(\varphi; \mathcal{A}, \mathcal{W})$ is integrable and to get classical orthogonality we have to make the following assumption:

Assumption 1.1. (a) If $\mathcal{R}(\varphi) = \sin((\varphi - \varphi_j)/2)^{k_j}\, \tilde{\mathcal{R}}(\varphi)$, then $\mathcal{W}(\varphi) = \sin((\varphi - \varphi_j)/2)^{m_j}\, \tilde{\mathcal{W}}(\varphi)$ where $m_j \geqslant (k_j - 1)/2$.

(b) $\partial\mathcal{A} - \partial\mathcal{W} + l/2 \in \mathbb{Z}$.

If assumption (b) is not fulfilled then one gets by the methods of this paper a non-classical orthogonality property of P_n with respect to $f(\varphi; \mathcal{A}, \mathcal{W})$ or to a functional of the form (1.17) below. Due to lack of space this case is not treated in this paper (the interested reader may see [28, Ch. 4]).

To get acquainted with the notation let us consider some illustrative examples for two arcs: Let

$$\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in [0, 2\pi) \qquad \text{with} \quad 0 \leqslant \varphi_1 < \varphi_2 < \varphi_3 < \varphi_4 < 2\pi$$

be given and let $\mathcal{R}(\varphi) = \sum_{k=0}^{2} a_k \cos k\varphi + b_k \sin k\varphi$ be the real trigonometric polynomial which vanishes exactly at the points φ_j, $j = 1, ..., 4$, and is negative on $Int(E_2)$, where

$$E_2 = [\varphi_1, \varphi_2] \cup [\varphi_3, \varphi_4], \tag{1.10}$$

i.e., $\mathscr{R}$ is of the form

$$\mathscr{R}(\varphi) = \prod_{j=1}^{4} \sin\left(\frac{\varphi - \varphi_j}{2}\right).$$

Now let us study for this special case of two intervals, or of two arcs $\Gamma_{E_2} = \{e^{i\varphi}: \varphi \in E_2\}$, what kind of weight functions fit into the class of weight functions treated in this paper:

EXAMPLE 1. Let $\mathscr{A}$ be a real trigonometric polynomial which has an even number of zeros in (φ_2, φ_3), i.e.,

$$\mathscr{A} \in \Pi \qquad \text{with} \quad \mathscr{A}(\varphi) > 0 \text{ on } E_2,$$

in particular we could choose $\mathscr{A} \equiv 1$.

(a) If we set

$$\mathscr{W}(\varphi) = 1$$

then the orthogonality condition (1.9) takes by (1.7) the form

$$\int_{\varphi_1}^{\varphi_2} e^{-ij\varphi} P_n(e^{i\varphi}) \frac{1}{\sqrt{-\prod_{j=1}^{4} \sin\left(\frac{\varphi - \varphi_j}{2}\right)}} \frac{d\varphi}{\mathscr{A}(\varphi)}$$

$$- \int_{\varphi_3}^{\varphi_4} e^{-ij\varphi} P_n(e^{i\varphi}) \frac{1}{\sqrt{-\prod_{j=1}^{4} \sin\left(\frac{\varphi - \varphi_j}{2}\right)}} \frac{d\varphi}{\mathscr{A}(\varphi)} = 0$$

$$\text{for} \quad j = 0, ..., \tilde{n} - 1, \tag{1.11}$$

where $\tilde{n} \geqslant n$. Note that $\tilde{n} > n$ is possible since the weight function $f(\varphi; \mathscr{A}, 1)$ has a sign change on $[0, 2\pi]$. Furthermore, since $\mathscr{W} \equiv 1$ and thus by (1.4)

$$\mathscr{V}(\varphi) = \mathscr{R}(\varphi),$$

the orthogonality condition (1.9) for $f(\varphi; \mathscr{A}, \mathscr{V})$ reads as follows:

$$\int_{\varphi_1}^{\varphi_2} e^{-ij\varphi} P_n(e^{i\varphi}) \frac{\sqrt{-\prod_{j=1}^{4} \sin\left(\frac{\varphi-\varphi_j}{2}\right)}}{\mathscr{A}(\varphi)} d\varphi$$

$$-\int_{\varphi_3}^{\varphi_4} e^{-ij\varphi} P_n(e^{i\varphi}) \frac{\sqrt{-\prod_{j=1}^{4} \sin\left(\frac{\varphi-\varphi_j}{2}\right)}}{\mathscr{A}(\varphi)} d\varphi = 0$$

$$\text{for } j=0, ..., \hat{n}-1, \hat{n} \geqslant n. \tag{1.12}$$

Naturally P_n from (1.11) and (1.12) will not be the same in general. Other choices of $\mathscr{W}$, which lead to weight functions having exactly one sign change, are obviously the following

$$\mathscr{W}(\varphi) = \sin\left(\frac{\varphi-\varphi_j}{2}\right), \qquad j \in \{1, 4\}$$

$$\mathscr{W}(\varphi) = \sin\left(\frac{\varphi-\varphi_2}{2}\right) \sin\left(\frac{\varphi-\varphi_3}{2}\right)$$

or

$$\mathscr{W}(\varphi) = \sin\left(\frac{\varphi-\varphi_1}{2}\right) \sin\left(\frac{\varphi-\varphi_4}{2}\right).$$

Next let us demonstrate how to get weight functions in the classical sense, i.e., weight functions nonnegative on E_2, which might be of most interest.

(b) If

$$\mathscr{W}(\varphi) = \sin\left(\frac{\varphi-\varphi_2}{2}\right), \qquad \mathscr{V}(\varphi) = \prod_{\substack{j=1 \\ j\neq 2}}^{4} \sin\left(\frac{\varphi-\varphi_j}{2}\right)$$

then $f(\varphi; \mathscr{A}, \mathscr{W})$ from (1.7) becomes a nonnegative function, more precisely the orthogonality relation (1.9) becomes

$$\int_{E_2} e^{-ij\varphi} P_n(e^{i\varphi})$$

$$\sqrt{\frac{\sin\left(\frac{\varphi-\varphi_2}{2}\right)}{\sin\left(\frac{\varphi-\varphi_3}{2}\right)}} \frac{1}{\sqrt{-\sin\left(\frac{\varphi-\varphi_1}{2}\right)\sin\left(\frac{\varphi-\varphi_4}{2}\right)}} \frac{d\varphi}{\mathscr{A}(\varphi)} = 0$$

$$\text{for } \quad j=0, ..., n-1,$$

and with respect to $-f(\varphi; \mathscr{A}, \mathscr{V})$, which is also a nonnegative weight function,

$$\int_{E_2} e^{-ij\varphi} P_n(e^{i\varphi}) \sqrt{\frac{\sin\left(\frac{\varphi-\varphi_3}{2}\right)}{\sin\left(\frac{\varphi-\varphi_2}{2}\right)}} \sqrt{-\sin\left(\frac{\varphi-\varphi_1}{2}\right)\sin\left(\frac{\varphi-\varphi_4}{2}\right)} \frac{d\varphi}{\mathscr{A}(\varphi)} = 0$$

for $j = 0, ..., n-1$.

Similarly, choosing

$$\mathscr{W}(\varphi) = \sin\left(\frac{\varphi-\varphi_2}{2}\right)\sin\left(\frac{\varphi-\varphi_4}{2}\right), \qquad \mathscr{V}(\varphi) = \sin\left(\frac{\varphi-\varphi_1}{2}\right)\sin\left(\frac{\varphi-\varphi_3}{2}\right)$$

(1.9) becomes for the nonnegative weight function $-f(\varphi; \mathscr{A}, \mathscr{W})$

$$\int_{E_2} e^{-ij\varphi} P_n(e^{i\varphi}) \sqrt{-\frac{\sin\left(\frac{\varphi-\varphi_2}{2}\right)\sin\left(\frac{\varphi-\varphi_4}{2}\right)}{\sin\left(\frac{\varphi-\varphi_1}{2}\right)\sin\left(\frac{\varphi-\varphi_3}{2}\right)}} \frac{d\varphi}{\mathscr{A}(\varphi)} = 0$$

for $j = 0, ..., n-1$

and for the nonnegative weight function $f(\varphi; \mathscr{A}, \mathscr{V})$

$$\int_{E_2} e^{-ij\varphi} P_n(e^{i\varphi}) \sqrt{-\frac{\sin\left(\frac{\varphi-\varphi_1}{2}\right)\sin\left(\frac{\varphi-\varphi_3}{2}\right)}{\sin\left(\frac{\varphi-\varphi_2}{2}\right)\sin\left(\frac{\varphi-\varphi_4}{2}\right)}} \frac{d\varphi}{\mathscr{A}(\varphi)} = 0$$

for $j = 0, ..., n-1$.

Other choices of $\mathscr{W}$ which lead to nonnegative weight functions are

$$\mathscr{W}(\varphi) = \sin\left(\frac{\varphi-\varphi_3}{2}\right)$$

$$\mathscr{W}(\varphi) = \sin\left(\frac{\varphi-\varphi_1}{2}\right)\sin\left(\frac{\varphi-\varphi_3}{2}\right),$$

$$\mathscr{W}(\varphi) = -\sin\left(\frac{\varphi-\varphi_2}{2}\right)\sin\left(\frac{\varphi-\varphi_4}{2}\right)$$

$$\mathscr{W}(\varphi)=\sin\left(\frac{\varphi-\varphi_1}{2}\right)\sin\left(\frac{\varphi-\varphi_2}{2}\right),$$

$$\mathscr{W}(\varphi)=-\sin\left(\frac{\varphi-\varphi_3}{2}\right)\sin\left(\frac{\varphi-\varphi_4}{2}\right).$$

EXAMPLE 2. (a) Now let $\mathscr{A}$ be a trigonometric polynomial which has exactly one sign change in (φ_2, φ_3) and $(\varphi_4, 2\pi)$ and no other zeros in $[0, 2\pi]$, i.e., $\mathscr{A}$ is of the form

$$\mathscr{A}(\varphi)=\sin\left(\frac{\varphi-\xi_1}{2}\right)\sin\left(\frac{\varphi-\xi_2}{2}\right)\tilde{\mathscr{A}}(\varphi), \tag{1.13}$$

where $\xi_1\in(\varphi_2, \varphi_3)$, $\xi_2\in(\varphi_4, 2\pi)$ and $\tilde{\mathscr{A}}(\varphi)>0$ on $[0, 2\pi]$. Then for

$$\mathscr{W}(\varphi)=1$$

the orthogonality condition (1.9) becomes

$$\int_{E_2} e^{-ij\varphi}P_n(e^{i\varphi})\frac{1}{|\mathscr{A}(\varphi)|\sqrt{-\prod_{j=1}^{4}\sin\left(\frac{\varphi-\varphi_j}{2}\right)}}\,d\varphi=0$$
$$\text{for}\quad j=0, ..., n-1. \tag{1.14}$$

Since $\mathscr{W}(\varphi)=1$ we have by (1.4)

$$\mathscr{V}(\varphi)=\mathscr{R}(\varphi)$$

and the orthogonality condition (1.9) for $f(\varphi; \mathscr{A}, \mathscr{V})$ becomes

$$\int_{E_2} e^{-ij\varphi}P_n(e^{i\varphi})\frac{\sqrt{-\prod_{j=1}^{4}\sin\left(\frac{\varphi-\varphi_j}{2}\right)}}{|\mathscr{A}(\varphi)|}\,d\varphi=0$$
$$\text{for}\quad j=0, ..., n-1. \tag{1.15}$$

Other nonnegative weight functions can be obtained by choosing

$$\mathscr{W}(\varphi)=\sin\left(\frac{\varphi-\varphi_1}{2}\right)\sin\left(\frac{\varphi-\varphi_4}{2}\right)$$

of

$$\mathscr{W}(\varphi)=-\sin\left(\frac{\varphi-\varphi_2}{2}\right)\sin\left(\frac{\varphi-\varphi_3}{2}\right).$$

(b) Let us note that the trigonometric polynomial $\mathscr{A}$ in (a) is of integer degree since $\tilde{\mathscr{A}}(\varphi) > 0$ on $[0, 2\pi]$. If we set

$$\mathscr{A}(\varphi) = \sin\left(\frac{\varphi - \xi_1}{2}\right)\tilde{\mathscr{A}}(\varphi),$$

$$\text{where} \quad \xi_1 \in (\varphi_2, \varphi_3) \qquad \text{and} \qquad \tilde{\mathscr{A}}(\varphi) > 0 \text{ on } [0, 2\pi], \tag{1.16}$$

where $\tilde{\mathscr{A}}$ is from Π, then $\mathscr{A}$ is of half integer degree and for

$$\mathscr{W}(\varphi) = 1, \qquad \mathscr{V}(\varphi) = \mathscr{R}(\varphi)$$

the orthogonality condition (1.9) looks the same as in (1.14) and (1.15), but Assumption 1.1(b) is hurt.

The reason why we considered not only the weight functions $f(\varphi; \mathscr{A}, \mathscr{W})$ but also $f(\varphi; \mathscr{A}, \mathscr{V})$ is that it will turn out that the polynomials orthogonal with respect to $f(\varphi; \mathscr{A}, \mathscr{W})$ and $f(\varphi; \mathscr{A}, \mathscr{V})$ are very closely related to each other, comparable to the Chebyshev polynomials of first and second kind on the real line.

Example 3. Next let us point out that the weight functions considered above multiplied by a real trigonometric polynomial $\mathscr{S}$ having all its zeros in $[0, 2\pi] \backslash Int(E_2)$, i.e., $\mathscr{S}$ is of the form

$$\mathscr{S}(\varphi) = c \prod_{j=1}^{k} \sin\left(\frac{\varphi - \psi_j}{2}\right), \qquad c \in \mathbb{R}\backslash\{0\}, \quad k \in \mathbb{N}, \quad \psi_j \notin Int(E_2),$$

are also covered in what follows (compare (1.25) below). For an example let us choose a real trigonometric polynomial $\mathscr{S}$ of degree one which has exactly one zero in each of the intervals $(0, \varphi_1)$ and $(\varphi_4, 2\pi)$, or exactly two signs changes in (φ_2, φ_3), i.e., $\mathscr{S}$ is of the form

$$\mathscr{S}(\varphi) = c \cdot \sin\left(\frac{\varphi - \psi_1}{2}\right) \sin\left(\frac{\varphi - \psi_2}{2}\right),$$

$$c \in \mathbb{R}\backslash\{0\}, \psi_1 \in (0, \varphi_1), \psi_2 \in (\varphi_4, 2\pi) \text{ or } \psi_1, \psi_2 \in (\varphi_2, \varphi_3).$$

Multiplying $\mathscr{S}$, say by the weight function from (1.15), we obtain an orthogonality condition of the form

$$\int_{E_2} e^{-ij\varphi} P_n(e^{i\varphi}) \frac{\mathscr{S}(\varphi)}{|\mathscr{A}(\varphi)|} \sqrt{-\prod_{j=1}^{4} \sin\left(\frac{\varphi - \varphi_j}{2}\right)}\, d\varphi = 0, \qquad j = 0, ..., n-1.$$

As already announced above, we investigate the more general case of polynomials orthogonal with respect to the linear functional

$$\mathscr{L}(h; \mathscr{A}, \mathscr{W}, \lambda) := \frac{1}{2\pi} \int_{E_l} h(e^{i\varphi})\, f(\varphi; \mathscr{A}, \mathscr{W})\, d\varphi + \mathscr{G}(h; \mathscr{A}, \mathscr{W}, \lambda) \tag{1.17}$$

with (recall the notation in (1.6))

$$\mathscr{G}(h; \mathscr{A}, \mathscr{W}, \lambda) := \frac{1}{2} \sum_{j=1}^{m^*} (1-\lambda_j) \sum_{v=0}^{m_j-1} \mu_{j,v} (-1)^v \, \delta_{z_j}^{(v)} \left(\frac{h(z)}{z} \right), \tag{1.18}$$

where the $\mu_{j,v}$'s are certain complex numbers depending on $\mathscr{A}$, $\mathscr{W}$ and $\mathscr{R}$ (for the exact description see theorem 2.1), $z_j := e^{i\xi_j} \in \mathbb{C} \backslash \Gamma_{E_l}$, $\delta_{z_j}^{(v)}(g) := (-1)^v \, g^{(v)}(z_j)/v!$ and $\lambda = (\lambda_1, ..., \lambda_{m^*}) \in \Delta_{m^*}$, where

$$\Delta_{m^*} := \left\{ (\lambda_1, ..., \lambda_{m^*}) : \lambda_j \in \{-1, 1\} \text{ and } \lambda_{j_1} = \lambda_{j_2} \text{ if } z_{j_1} = \frac{1}{\overline{z_{j_2}}} \right\} \tag{1.19}$$

(note that $z_{j_1} = 1/\overline{z_{j_2}}$ is equivalent with $\xi_{j_1} = \overline{\xi_{j_2}}$), i.e., if $\lambda_j = -1$ then there appears a “Dirac mass-point” at $z_j = e^{i\xi_j}$.

Hence we are describing those polynomials P_n of degree n which satisfy the orthogonality condition

$$\begin{aligned} \mathscr{L}(z^{-k} P_n; \mathscr{A}, \mathscr{W}, \lambda) = {} & \frac{1}{2\pi} \int_{E_l} e^{-ik\varphi} P_n(e^{i\varphi}) \, f(\varphi; \mathscr{A}, \mathscr{W}) \, d\varphi \\ & + \frac{1}{2} \sum_{j=1}^{m^*} (1-\lambda_j) \sum_{v=0}^{m_j-1} \mu_{j,v} (-1)^v \, \delta_{z_j}^{(v)} (z^{-(k+1)} P_n) = 0 \\ & \text{for} \quad k = 0, ..., n-1. \end{aligned} \tag{1.20}$$

We call a functional $\mathscr{L}$ positive-definite resp. definite if $\det(c_{j-k})_{j,k=0}^n > 0$ resp. $\neq 0$ for all $n \in \mathbb{N}_0$, where the moments c_j are given by $c_j := \mathscr{L}(z^{-j})$, $j \in \mathbb{Z}$. Note that we *don't* suppose the linear functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ to be positive-definite and also not to be definite.

For an illustrative example where point measures appear see example (d) below. First let us list in (a)–(c) those weight functions which appear as special cases of our functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ with $(\lambda_1, ..., \lambda_{m^*}) = (1, 1, ..., 1)$, i.e., $\mathscr{G}(\cdot; \mathscr{A}, \mathscr{W}, \lambda) \equiv 0$, and which seem to be the most important and interesting ones.

EXAMPLES. (a) Suppose that the trigonometric polynomial $\mathscr{A}$ has no zeros in $[0, 2\pi]$. Then

$$\begin{aligned} f(\varphi; \mathscr{A}, 1) &= \frac{(-1)^j}{\mathscr{A}(\varphi) \sqrt{-\prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_j}{2}\right)}} \qquad \text{and} \\ f(\varphi; \mathscr{A}, \mathscr{R}) &= \frac{(-1)^j \sqrt{-\prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_j}{2}\right)}}{\mathscr{A}(\varphi)} \end{aligned} \tag{1.21}$$

for $\varphi \in [\varphi_{2j-1}, \varphi_{2j}]$, $j=1, ..., l$, and

$$f(\varphi;\mathscr{A},1)=f(\varphi;\mathscr{A},\mathscr{R})=0 \qquad \text{for} \quad \varphi\notin E_l.$$

Note that $f(\varphi;\mathscr{A},1)$ and $f(\varphi;\mathscr{A},\mathscr{R})$ change sign from arc to arc.

Multiplying $f(\varphi;\mathscr{A},1)$ by

$$\mathscr{W}(\varphi)=\prod_{j=1}^{l}\sin\left(\frac{\varphi-\varphi_{2j-1}}{2}\right), \qquad \mathscr{V}(\varphi)=\prod_{j=1}^{l}\sin\left(\frac{\varphi-\varphi_{2j}}{2}\right) \tag{1.22}$$

we obtain the non sign-changing weight functions

$$\left.\begin{aligned} f(\varphi;\mathscr{A},\mathscr{W})&=\frac{(-1)^l}{\mathscr{A}(\varphi)}\sqrt{(-1)^l\frac{\prod_{j=1}^{l}\sin\left(\frac{\varphi-\varphi_{2j-1}}{2}\right)}{\prod_{j=1}^{l}\sin\left(\frac{\varphi-\varphi_{2j}}{2}\right)}}\\ f(\varphi;\mathscr{A},\mathscr{V})&=\frac{(-1)^l}{\mathscr{A}(\varphi)}\sqrt{(-1)^{l+1}\frac{\prod_{j=1}^{l}\sin\left(\frac{\varphi-\varphi_{2j}}{2}\right)}{\prod_{j=1}^{l}\sin\left(\frac{\varphi-\varphi_{2j-1}}{2}\right)}}\end{aligned}\right\} \quad \varphi\in E_l \tag{1.23}$$

and

$$f(\varphi;\mathscr{A},\mathscr{W})=f(\varphi;\mathscr{A},\mathscr{V})=0 \qquad \text{for} \quad \varphi\notin E_l.$$

(b) If $\mathscr{A}$ has an odd number of zeros in each interval $(\varphi_{2j}, \varphi_{2j+1})$, $j=1, ..., l-1$, we get the non sign-changing weight functions, $\varphi\in E_l$,

$$f(\varphi;\mathscr{A},1)=\frac{1}{|\mathscr{A}(\varphi)|\sqrt{-\prod_{j=1}^{2l}\sin\left(\frac{\varphi-\varphi_j}{2}\right)}}$$

and (1.24)

$$f(\varphi;\mathscr{A},\mathscr{R})=\frac{\sqrt{-\prod_{j=1}^{2l}\sin\left(\frac{\varphi-\varphi_j}{2}\right)}}{|\mathscr{A}(\varphi)|}$$

and

$$f(\varphi;\mathscr{A},1)=f(\varphi;\mathscr{A},\mathscr{R})=0 \qquad \text{for} \quad \varphi\notin E_l.$$

Of course, multiplying $f(\varphi; \mathscr{A}, 1)$ by $\mathscr{W}$ from (1.22) we would obtain a weight function changing sign from arc to arc.

(c) Let $\mathscr{S}$ be a trigonometric polynomial having all its zeros in $[0, 2\pi]\backslash Int(E_l)$ and having no common zeros with $\mathscr{A}$, i.e.,

$$\mathscr{S}(\varphi) = c \prod_{j=1}^{k} \sin\left(\frac{\varphi - \psi_j}{2}\right),$$
$$c \in \mathbb{R}\backslash\{0\},\ k \in \mathbb{N},\ k \text{ even},\ \psi_j \notin Int(E_l) \text{ and } \psi_j \notin \{\xi_1, ..., \xi_{m^*}\} \quad (1.25)$$

and let (note that we did not assume that $\mathscr{R}$ and $\mathscr{W}$ have to have simple zeros)

$$\tilde{\mathscr{R}}(\varphi) = \mathscr{S}^2(\varphi) \prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_j}{2}\right) \qquad \text{and} \qquad \tilde{E} = E_l \cup \{\psi_1, ..., \psi_k\}.$$

Let $\tilde{f}$ be the weight function from (1.7) associated with $\tilde{\mathscr{R}}$. Then for $\varphi \in [\varphi_{2j-1}, \varphi_{2j}]$, $j = 1, ..., l$,

$$\tilde{f}(\varphi; \mathscr{A}, \mathscr{S}^2) = \frac{(-1)^{j+1} \mathscr{S}(\varphi)}{\mathscr{A}(\varphi) \sqrt{-\prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_j}{2}\right)}}$$

and

$$\tilde{f}(\varphi; \mathscr{A}, \mathscr{S}^2\mathscr{R}) = \frac{(-1)^{j+1} \mathscr{S}(\varphi) \sqrt{-\prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_j}{2}\right)}}{\mathscr{A}(\varphi)}$$

and

$$\tilde{f}(\varphi; \mathscr{A}, \mathscr{S}^2) = \tilde{f}(\varphi; \mathscr{A}, \mathscr{S}^2\mathscr{R}) = 0 \qquad \text{for} \quad \varphi \notin E_l.$$

If we choose $\mathscr{W}$ as in (1.22) then

$$\tilde{f}(\varphi; \mathscr{A}, \mathscr{S}^2\mathscr{W}) = \frac{\mathscr{S}(\varphi)}{\mathscr{A}(\varphi)} \sqrt{(-1)^l \frac{\prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_{2j-1}}{2}\right)}{\prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_{2j}}{2}\right)}}$$

$$\text{for} \quad \varphi \in E_l \text{ and zero otherwise.}$$

(d) Suppose that $\mathscr{A}$ has only real zeros and exactly one simple zero in each interval $(\varphi_{2j}, \varphi_{2j+1})$, $j=1, ..., l$, $\varphi_{2l+1} := \varphi_1 + 2\pi$, i.e.,

$$\mathscr{A}(\varphi) = \prod_{j=1}^{l} \sin\left(\frac{\varphi - \xi_j}{2}\right), \qquad \text{where} \quad \xi_j \in (\varphi_{2j}, \varphi_{2j+1}).$$

Then for the weight $f(\varphi; \mathscr{A}, \mathscr{R})$ (see the second relation in (1.24)) the orthogonality condition (1.20) takes the form, by inserting the explicit expressions for $\mu_{j,0}$ (compare theorem 2.1 below),

$$\frac{1}{2\pi} \int_{E_l} e^{-ik\varphi} P_n(e^{i\varphi}) \frac{\sqrt{-\prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_j}{2}\right)}}{|\mathscr{A}(\varphi)|} \, d\varphi$$

$$+ \frac{1}{2} \sum_{j=1}^{l} (1 - \lambda_j) \frac{\sqrt{R(e^{i\xi_j})}}{i\left(\frac{d}{dz} A\right)(e^{i\xi_j})} e^{-i(k+1)\,\xi_j} P_n(e^{i\xi_j}) = 0$$

$$\text{for} \quad k = 0, ..., n-1, \tag{1.26}$$

where $R(e^{i\varphi}) := e^{il\varphi} \mathscr{R}(\varphi)$ and $A(e^{i\varphi}) := e^{i(l/2)\varphi} \mathscr{A}(\varphi)$. Recall that $\lambda_1, ..., \lambda_l$ can be chosen arbitrary from $\{-1, +1\}$. Relation (1.26) represents an orthogonality relation for P_n with respect to a positive measure $d\sigma$ which has mass points at those $e^{i\xi_j}$ where $\lambda_j = -1$.

Stated in a more general form: if $f(\varphi; \mathscr{A}, \mathscr{W})$ is a nonnegative weight function of the form (1.7) and if all real zeros $\xi_1, ..., \xi_{p^*}$, $p^* \leqslant m^*$, of $\mathscr{A}$ are simple and if $\lambda_j = 1$ for the zeros $\xi_j \notin \mathbb{R}$, then the orthogonality condition (1.20) represents an orthogonality relation for P_n with respect to a measure $d\sigma$ which has mass points at the ξ_j's, $j = 1, ..., p^*$. More precisely (1.20) becomes

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} P_n(e^{i\varphi}) \, d\sigma(\varphi; \mathscr{A}, \mathscr{W}, \lambda) = 0, \qquad k = 0, ..., n-1,$$

where with the same notation as above

$$d\sigma(\varphi; \mathscr{A}, \mathscr{W}, \lambda) := f(\varphi; \mathscr{A}, \mathscr{W}) \, d\varphi - \pi \sum_{j=1}^{p^*} (1 - \lambda_j)$$

$$\times \frac{i\sqrt{R(e^{i\xi_j})}}{\left(\frac{d}{dz} A\right)(e^{i\xi_j})} e^{-i\xi_j} \delta(e^{i\varphi} - e^{i\xi_j}) \, d\varphi.$$

Finally let us show what kind of weight functions, which appear as special cases of our functional, have been investigated in the literature so far. To get the single interval case, i.e., the case of the whole unit circumference, let $E_1 = [0, 2\pi]$ and let $\mathscr{R}(\varphi) = \sin(\varphi/2)\sin((\varphi+2\pi)/2) = -\sin^2(\varphi/2)$, then in view of Assumption 1.1 the following weight functions are covered

$$f\left(\varphi; \mathscr{A}, -\sin\frac{\varphi}{2}\right) = \frac{1}{\mathscr{A}(\varphi)},$$

where $\mathscr{A}$ is an arbitrary positive trigonometric polynomial of integer degree, i.e., $\mathscr{A}$ has no zeros in $[0, 2\pi]$. Thus $\mathscr{A}$ can be represented in the form $\mathscr{A}(\varphi) = |T(e^{i\varphi})|^2$, $\varphi \in [0, 2\pi]$, where $T(z)$ is an algebraic polynomial with all its zeros in $|z| < 1$. This means (see e.g., [31, p. 31] and [4]) that $f(\varphi; \mathscr{A}, -\sin\varphi/2)$ is the well-known Bernstein–Szegő weight function. Polynomials orthogonal with respect to Bernstein–Szegő weights play a central role in the asymptotic representation of orthogonal polynomials (see, e.g. [15]). For the simplest case $\mathscr{A} \equiv 1$ we obtain the Lebesgue measure and in this case, as it is well known, the orthogonal polynomials are of the form $P_n(z) = z^n$.

The other case which has been investigated from different points of view, like asymptotics etc., (see [1, 7, 8, 14]) is the case of an arc not necessarily the whole unit circumference, i.e.,

$$E_1 = [\alpha, 2\pi - \alpha], \quad \alpha \in (0, \pi), \qquad \mathscr{R}(\varphi) = -\sin\left(\frac{\varphi-\alpha}{2}\right)\sin\left(\frac{\varphi+\alpha}{2}\right).$$

In this setting (1.7) takes the form

$$f\left(\varphi; \mathscr{A}, \sin\left(\frac{\varphi+\alpha}{2}\right)\right) = \begin{cases} \sqrt{\dfrac{\sin\left(\frac{\varphi+\alpha}{2}\right)}{\sin\left(\frac{\varphi-\alpha}{2}\right)}} \cdot \dfrac{1}{\mathscr{A}(\varphi)}, & \text{for} \quad \varphi \in E_1 \\ 0, & \text{elsewhere}, \end{cases} \tag{1.27}$$

and with $\mathscr{A}(\varphi) = \sin((\varphi-\beta)/2)\,\tilde{\mathscr{A}}(\varphi)$, $\beta \in [-\alpha, \alpha]$,

$$f(\varphi; \mathscr{A}, \mathscr{R}) = \sqrt{\frac{\sin\left(\frac{\varphi-\alpha}{2}\right)\sin\left(\frac{\varphi+\alpha}{2}\right)}{\sin\left(\frac{\varphi-\beta}{2}\right)}} \cdot \frac{1}{\tilde{\mathscr{A}}(\varphi)}$$

$$\text{for} \quad \varphi \in E_1 \text{ and } 0 \text{ elsewhere.} \tag{1.28}$$

In fact in [1, 14] asymptotics for polynomials orthogonal with respect to the more general class of weight functions, when $\mathscr{A}$ is replaced in (1.27) and (1.28) by a continuous positive weight function on $[\alpha, 2\pi-\alpha]$, are obtained. But the methods used there seem to be limited to the one arc case.

It is also worth mentioning that Geronimus [7, 8, 9] has shown that polynomials with constant reflection coefficients are orthogonal to measures whose absolutely continuous part is of the form (1.28) and, more general, polynomials with periodic reflection coefficients are orthogonal to measures whose absolutely continuous part is of the form (1.24).

Let us briefly give an outline, for the simplest cases, of the main input of this paper. Loosely formulated, we shall show that a polynomial $P_n(z)= z^n+\ldots$ is orthogonal with respect to a functional of the form $\mathscr{L}(\cdot\,; \mathscr{A}, \mathscr{W}, \lambda)$ if and only if there exists a polynomial Q_m and a polynomial $g_{(n)} \in \mathbb{P}^{\mathbb{C}}_{l-1}$ such that

$$W(z)\, P_n^2(z) - V(z)\, Q_m^2(z) = z^n A(z)\, g_{(n)}(z). \tag{1.29}$$

Further, if (1.29) holds, then the polynomial Q_m can be represented in terms of P_n and the polynomial of the second kind and what is important, Q_n is orthogonal with respect to $f(\varphi; \mathscr{A}, \mathscr{V})$. In a forthcoming paper [25] we shall see that the polynomials $g_{(n)}$, $n \in \mathbb{N}$, which are all of degree less than or equal to $l-1$, contain almost all information on the polynomials orthogonal with respect to $\mathscr{L}(\cdot\,; \mathscr{A}, \mathscr{W}, \lambda)$ and also with respect to $\mathscr{L}(\cdot\,; \mathscr{A}, \mathscr{V}, \lambda)$. So it will turn out that the nth associated polynomials of the polynomials orthogonal with respect to $\mathscr{L}(\cdot\,; \mathscr{A}, \mathscr{W}, \lambda)$ are orthogonal with respect to a measure which is again of the type treated in this paper and can be described with the help of $g_{(n)}$. Furthermore the $g_{(n)}$'s are related to each other by a nonlinear recurrence relation from which a nonlinear recurrence relation for the reflection coefficients of the orthogonal polynomials can be extracted.

Let us demonstrate by another consideration the usefulness of (1.29). Suppose we know polynomials $\mathscr{T}_N$ and $\mathscr{U}_{N-l}$ satisfying a relation of the form

$$\mathscr{T}_N^2(z) - R(z)\, \mathscr{U}_{N-l}^2(z) = cz^N, \qquad c \in \mathbb{R}\backslash\{0\}, \tag{1.30}$$

then by multiplying (1.29) and (1.30) one gets by straightforward calculation

$$W(\mathscr{T}_N P_n + V\mathscr{U}_{N-l} Q_m)^2 - V(\mathscr{T}_N Q_m + W\mathscr{U}_{N-l} P_n)^2 = cz^{nN} A g_{(n)}. \tag{1.31}$$

Repeating this procedure to (1.30) and (1.31) we obtain an infinite sequence of orthogonal polynomials and in this way we can generate all orthogonal polynomials with respect to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ and simultaneously with respect to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{V}, \lambda)$. As a consequence of (1.31), and of the above characterization, one obtains $g_{(n+N)} = cg_{(n)}$, which gives periodicity of the reflection coefficients. Or in other words, a relation of the form (1.30) is fulfilled if and only if the reflection coefficients of the polynomials orthogonal with respect to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ are periodic from a certain index onward. For the rigorous proofs of these and other facts see the authors forthcoming papers [25, 26].

Why is it important to know polynomials orthogonal with respect to $f(\varphi; \mathscr{A}, \mathscr{W})$ or $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ besides being of interest in its own right? One of the reasons is that when we have information on the polynomials orthogonal with respect to $f(\varphi; \mathscr{A}, \mathscr{W})$ then we know the asymptotic behaviour of the orthogonal polynomials for such classes of weight functions which can be sufficiently well approximated by weight functions of the form $f(\varphi; \mathscr{A}, \mathscr{W})$, that is, for which a sequence of trigonometric polynomials $\mathscr{A}_n$ can be constructed such that the $f(\varphi; \mathscr{A}_n, \mathscr{W})$'s converge to the desired weight function on E_l. By the way, let us point out in this connection that $\mathscr{A}$ is only assumed to have all zeros outside E_l and that there is no restriction on the size of the degree of $\mathscr{A}$. It's worth mentioning that this was exactly the way how Szegő [31] obtained his famous asymptotic formula for polynomials orthogonal with respect to weight functions from the Szegő-class. The main advantage in the Szegő case, i.e., the case of the whole unit circumference, is that the polynomials orthogonal with respect to $f(\varphi; \mathscr{A}, \mathscr{W})$ can easily be determined (see [4, 31]) in contrast to the several arc case. As we have demonstrated above, in the several arc case an explicit determination is easily possible if there exists a polynomial $\mathscr{T}_N$ on E_l satisfying relation (1.30) or in other words if the polynomials orthogonal with respect to $f(\varphi; \mathscr{A}, \mathscr{W})$ resp. $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ have periodic reflection coefficients. Applying the above described procedure of approximation we will get asymptotic formulas for orthogonal polynomials having asymptotically periodic reflection coefficients. Details will be given in [25, 26]. For the general case asymptotic representations can be obtained by assuming that the reflection coefficients behave asymptotically like those of the nth, n large, associated polynomials. Considerations of this kind are under work.

Polynomials on several real intervals, resp. several arcs—recall that polynomials orthogonal on the unit circle lead to polynomials orthogonal on the interval $[-1, 1]$—play also an important role in modern solid state physics [5, 16, 27, 29, 30] because in problems of this field it's natural that the densities of states have band structure, i.e., live on several arcs, resp. several intervals. For the mathematical approach to the case of several real intervals compare [2, 3, 11, 12, 13, 18–20]. In this paper we try to carry over some of the main ideas of [20] to the several arc case, where the situation turns out to be much more complicated than the real case. It is surprising that not much is known about orthogonal polynomials on several arcs. As already mentioned the only results on this topic come from Geronimus [6–9] and Achieser [1] and a very recent contribution on polynomials with asymptotically constant recurrence coefficients from Golinskii, Nevai and Van Assche [14].

In order to be able to state our results we need some preliminaries and notation. Let us first state the well known relation between algebraic and trigonometric polynomials: Let $\mathscr{D} \in \Pi$ of degree $\partial \mathscr{D} = n/2$, then we assign to $\mathscr{D}$ the algebraic polynomial $D \in \mathbb{P}_n^{\mathbb{C}}$ given by

$$D(e^{i\varphi}) := e^{i(n/2)\varphi} \mathscr{D}(\varphi) = e^{i\,\partial \mathscr{D} \varphi} \mathscr{D}(\varphi). \tag{1.32}$$

Hence $\partial D = n = 2\partial \mathscr{D}$ and D is a selfreciprocal polynomial, i.e., $D = D^*$, where the *reciprocal* polynomial D^* is defined by

$$D^*(z) := z^n \bar{D}\left(\frac{1}{z}\right) = z^{\partial D} \bar{D}\left(\frac{1}{z}\right). \tag{1.33}$$

On the other hand each polynomial $D \in \mathbb{P}^{\mathbb{C}}$ of degree $\partial D = n$, satisfying $D = D^*$, induces a real trigonometric polynomial $\mathscr{D}$ by

$$\mathscr{D}(\varphi) := e^{-i(n/2)\varphi} D(e^{i\varphi}) \qquad \text{with} \quad \partial \mathscr{D} = \frac{n}{2}.$$

Besides (1.33) we'll also need the following notation: if $C \in \mathbb{P}_m^{\mathbb{C}}$ is a polynomial of exact degree $\partial C \leqslant m$, we define the *modified reciprocal* polynomial $C_m^{(*)}$ by

$$C_m^{(*)}(z) := z^m \bar{C}\left(\frac{1}{z}\right) = z^{m-\partial C} C^*(z). \tag{1.34}$$

Notice that the exponent m of z in (1.34) is equal to the subindex on the left hand side and that for the modified reciprocal polynomial the index m must be written explicitly. The reason why we distinguish between the modified reciprocal polynomial $C_m^{(*)}$ and the reciprocal polynomial C^* is that $C_m^{(*)}(0) = 0$ is possible (if $\partial C < m$) whereas $C^*(0)$ is always different from zero.

We finish this section with the following additional notation:

(a) The polynomial of the second kind with respect to a linear functional $\mathscr{L}$ (not necessarily of the form (1.17)) is defined by

$$\Omega_n(z) := \begin{cases} \mathscr{L}\left(\dfrac{x+z}{x-z}\,(P_n(x)-P_n(z))\right), & \text{if} \quad n \in \mathbb{N} \\ \mathscr{L}(1)\,P_0(0), & \text{if} \quad n=0, \end{cases} \tag{1.35}$$

where $\mathscr{L}$ acts on x. We see that $\partial\Omega_n = \max\{-1, n-k\}$ if $c_0 = \cdots = c_{k-1} = 0$, $c_k \neq 0$. Here the c_j's are given by the power series expansion $\mathscr{L}((x+z)/(x-z)) = c_0 + 2\sum_{j=1}^{\infty} c_j z^j$.

(b) Let $P_n \in \mathbb{P}_n^{\mathbb{C}}$ be an arbitrary polynomial and let $\kappa, \mu \in \mathbb{Z}$. The orthogonality property

$$\mathscr{L}(z^{-j}P_n) = 0, \qquad j = \kappa+1, ..., \mu-1,$$
$$\mathscr{L}(z^{-\kappa}P_n) \neq 0, \qquad \text{and} \qquad \mathscr{L}(z^{-\mu}P_n) \neq 0$$

will be abbreviated by

$$\mathscr{L}(z^{-j}P_n) = 0, \qquad j \in [\kappa+1, ..., \mu-1],$$

and we use round brackets "(" resp. ")", if the lower order $\kappa+1$ resp. the upper order $\mu-1$ is not necessarily the maximal one.

(c) Let the function H be analytic at $z=0$, then we write $H(z) = O(z^\nu)$, $\nu \in \mathbb{N}_0$, if H has a representation of the form $H(z) = \sum_{j=\nu}^{\infty} K_j z^j$ for $|z|$ sufficiently small. If $K_\nu \neq 0$, we write $H(z) = \dot{O}(z^\nu)$.

This paper is organized as follows. In Section 2 we state the main results which are the explicit representation of the Stieltjes transform of the functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$. A characterization of the orthogonal polynomials by a quadratic equation and an integral representation of the orthogonal polynomials and of the associated polynomials is given. In Section 3 some preliminary results are given, where in particular the square root function $\sqrt{R}$ is discussed in detail, which is needed for the proof of Theorem 2.1. Furthermore another characterization of orthogonal polynomials is given, which is the basis for the above mentioned characterization by the quadratic equation. Finally, in Section 4 the main results are proved.

2. Statement of the Main Results

In order to be able to state our results in closed form let

$$v := \partial\mathscr{V}, \qquad w := \partial\mathscr{W}, \qquad a := \max\left\{\partial\mathscr{A}, w - \frac{l}{2}\right\} \tag{2.1}$$

and define the algebraic polynomials R, V, W and A by

$$\begin{aligned} R(e^{i\varphi}) &:= e^{il\varphi}\mathscr{R}(\varphi), & \partial R &= 2l \\ V(e^{i\varphi}) &:= e^{iv\varphi}\mathscr{V}(\varphi), & \partial V &= 2v \\ W(e^{i\varphi}) &:= e^{iw\varphi}\mathscr{W}(\varphi), & \partial R &= 2w \\ A(e^{i\varphi}) &:= e^{ia\varphi}\mathscr{A}(\varphi), & \partial A &= a + \partial\mathscr{A}, \end{aligned} \tag{2.2}$$

where

$$A(z) = z^{a-\partial\mathscr{A}}\tilde{A}(z) \qquad \text{with} \quad \tilde{A}(e^{i\varphi}) := e^{i\,\partial\mathscr{A}\,\varphi}\mathscr{A}(\varphi).$$

In view of (1.6) we have the explicit representation

$$A(z) = c_A z^{a-\partial\mathscr{A}} \prod_{j=1}^{m^*} (z - z_j)^{m_j}, \qquad \text{where} \quad c_A \in \mathbb{C}, \quad z_j = e^{i\xi_j}. \tag{2.3}$$

Let us mention that by (2.1) $a = \partial\mathscr{A}$ if $\partial\mathscr{A} \geqslant w - l/2$, which will be fulfilled in most of the interesting cases. So the reader, who is not interested in details, should associate with a the degree of $\mathscr{A}$. By the definitions (2.2) the above polynomials are selfreciprocal; to be more precise we have $R = R^*$, $V = V^*$, $W = W^*$ and $A = A_{2a}^{(*)}$.

For the following let us point out that we choose that branch of the square root of $\sqrt{R}$, which is analytic on $\mathbb{C}\backslash\Gamma_{E_l}$ and satisfies (compare Lemma 3.1 and Remark 3.2 in Section 3)

$$\begin{aligned} \operatorname{sgn}\sqrt{R(e^{i\varphi})} = (-1)^j\, e^{i(l/2)\varphi} \qquad &\text{for} \quad \varphi \in (\varphi_{2j}, \varphi_{2j+1}), \\ j = 0, ..., l, \qquad &\varphi_{2l+1} := \varphi_1 + 2\pi. \end{aligned} \tag{2.4}$$

One of the main reasons that many properties about polynomials orthogonal with respect to weight functions of the form (1.7) or with respect to a functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ of the form (1.17) can be derived is that the Stieltjes transform of the functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ can be given explicitly as our first theorem shows.

THEOREM 2.1. *Let A, R, V, W and the vector $\lambda \in \Delta_{m^*}$ be given and let $B := B(\cdot; A, W, \lambda) \in \mathbb{P}_{2a}^{\mathbb{C}}$ be the uniquely determined selfreciprocal polynomial $B = B_{2a}^{(*)}$, which satisfies (compare with Lemma 3.3 below) the interpolation conditions*

$$(VB)^{(\nu)}(z_j) = -\lambda_j (z^{a+l/2-w}\sqrt{R})^{(\nu)}(z_j), \qquad \nu = 0, ..., m_j - 1, \quad j = 1, ..., m^*$$

and the "zero-condition"

$$\lim_{z \to 0} \frac{B(z) + z^{a+l/2-w} \dfrac{W(z)}{\sqrt{R(z)}}}{iA(z)} \in \mathbb{R}.$$

Then the following identity holds for $z \in \mathbb{C} \setminus (\Gamma_{E_l} \cup \{z_j : \lambda_j = -1\})$, *where we use the same notation as in* (1.17):

$$\begin{aligned} \mathscr{L}\left(\frac{x+z}{x-z}; \mathscr{A}, \mathscr{W}, \lambda\right) &= \frac{1}{2\pi} \int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \frac{\mathscr{W}(\varphi)}{\mathscr{A}(\varphi)\, r(\varphi)}\, d\varphi \\ &\quad + \frac{1}{2} \sum_{j=1}^{m^*} (1-\lambda_j) \sum_{v=0}^{m_j-1} \mu_{j,v} (-1)^v \, \delta_{z_j}^{(v)} \left(\frac{1}{x} \frac{x+z}{x-z}\right) \\ &= -\frac{B(z) + z^{a+l/2-w} \dfrac{W(z)}{\sqrt{R(z)}}}{iA(z)}; \end{aligned} \tag{2.5}$$

here $\delta_{z_j}^{(v)}$ *acts on* x *and the constants* $\mu_{j,v}$ *are given by*

$$\mu_{j,v} := \frac{1}{(m_j-1-v)!} \left(\frac{z^{a+l/2-w} W}{iA_j \sqrt{R}}\right)^{(m_j-1-v)} (z_j),$$

$$A_j(z) := \frac{A(z)}{(z-z_j)^{m_j}}.$$

If the functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ is positive definite then $\mathscr{L}((x+z)/(x-z); \mathscr{A}, \mathscr{W}, \lambda)$ is a Caratheodory-function.

From the definition of the $\mu_{j,v}$'s one obtains that the linear functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ from (1.17) can be written in the compact form

$$\begin{aligned} \mathscr{L}(h; \mathscr{A}, \mathscr{W}, \lambda) &= \frac{1}{2\pi} \int_{E_l} h(e^{i\varphi})\, f(\varphi; \mathscr{A}, \mathscr{W})\, d\varphi \\ &\quad + \frac{1}{2} \sum_{j=1}^{m^*} \frac{1-\lambda_j}{(m_j-1)!} \left(\frac{z^{a+l/2-w-1} Wh}{iA_j \sqrt{R}}\right)^{(m_j-1)} (z_j). \end{aligned} \tag{2.6}$$

The following theorem is the main result of this paper and characterizes polynomials orthogonal with respect to a weight function of the form (1.7) or to a functional of the form (1.17).

THEOREM 2.2. *Let the polynomials* $R, V, W, A, B := B(\cdot; A, W, \lambda)$ *and the linear functional* $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ *be given. Let* $\mu \in \mathbb{N}_0$ *and* P_n *be a polynomial of degree* n, *where* $n \geqslant \max\{a + l/2 - w, 2v - l\}$ *if* $\mu > 0$ *and*

$n \geqslant \max\{a + l/2 - w, 2v - l, \partial\mathscr{A} + l/2 + v\}$ *if* $\mu = 0$. *Furthermore let* p *be the multiplicity of the zero of* P_n *at* $z = 0$. *Finally if the polynomials* P_n *and* $A/z^{a-\partial\mathscr{A}}$ *have a common zero at* z_j, *then* z_j *is supposed to be a simple zero of* P_n *and of* A. *Then the following propositions are equivalent*:

(a) $\mathscr{L}(z^{-j}P_n; \mathscr{A}, \mathscr{W}, \lambda) = 0$ *for* $j \in (0, ..., n + \mu - 1]$ *and there exists a* $\tau \in \mathbb{N}$ *such that* $\mathscr{L}(z^{\tau}P_n; \mathscr{A}, \mathscr{W}, \lambda) \neq 0$.

(b) *There exists a polynomial* $Q_{n+l-2v} \in \mathbb{P}^{\mathbb{C}}_{n+l-2v}$ *and there exists a polynomial* $g_{(n)}$ *with* $g_{(n)}(0) \neq 0$ *from* $\mathbb{P}^{\mathbb{C}}_{l-2\mu}$ *if* $\mu > 0$ *and from* $\mathbb{P}^{\mathbb{C}}_{l-p-1}$ *if* $\mu = 0$, *such that*

$$W(z)\, P_n^2(z) - V(z)\, Q_{n+l-2v}^2(z) = z^{n+p-(a+l/2-w)+\mu} A(z)\, g_{(n)}(z), \tag{2.7}$$

where $p \leqslant \mu - 1$ *if* $\mu > 0$, *and*

$$V(z_j)\, Q_{n+l-2v}(z_j) = \lambda_j \sqrt{R(z_j)}\, P_n(z_j), \qquad j = 1, ..., m^*, \tag{2.8}$$

$$\left.\frac{VQ_{n+l-2v}}{\sqrt{R}\, P_n}\right|_{z=0} = 1, \qquad V(0)\, Q^*_{n+l-2v}(0) = \sqrt{R(0)}\, P_n^*(0). \tag{2.9}$$

(c) *There exists a polynomial* $g_{(n)}$ *with* $g_{(n)}(0) \neq 0$ *from* $\mathbb{P}^{\mathbb{C}}_{l-2\mu}$ *if* $\mu > 0$ *and from* $\mathbb{P}^{\mathbb{C}}_{l-p-1}$ *if* $\mu = 0$, *such that*

$$\begin{aligned} W(z)\, P_n^2(z) - V(z) &\left(\frac{i\Omega_n(z)\, A(z) - P_n(z)\, B(z)}{z^{a+l/2-w}}\right)^2 \\ &= z^{n+p-(a+l/2-w)+\mu} A(z)\, g_{(n)}(z), \end{aligned} \tag{2.10}$$

where $p \leqslant \mu - 1$ *if* $\mu > 0$, *and* ("*sign-condition*")

$$\left(\frac{\Omega_n}{P_n}\right)(z) - \frac{B(z) + z^{a+l/2-w} \dfrac{W(z)}{\sqrt{R(z)}}}{iA(z)} = O(z^{\max\{0, \partial\mathscr{A} + l/2 - w + 1\}}). \tag{2.11}$$

In (2.10) *resp.* (2.11) Ω_n *denotes the polynomial of the second kind with respect to* $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$.

Thus, Theorem 2.2 gives the following result: given a polynomial P_n, in order to decide if it is orthogonal with respect to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ one only has to look for another polynomial Q_{n+l-2v} (and this polynomial is of the form given in (2.10) if it exists) and to calculate the polynomial expression on the left hand side of (2.7). Then P_n is orthogonal if and only if the left hand side of (2.7) has a zero at $z = 0$ of multiplicity $n + p - (\partial\mathscr{A} + l/2 - w) + \mu$ and a degree $\leqslant n + \partial\mathscr{A} + l/2 + w - 1$ if $\mu = 0$ resp. $\leqslant n + p + \partial\mathscr{A} + l/2 + w - \mu$ if $\mu > 0$ and the "sign-conditions" (2.8) and (2.9) are fulfilled.

Remark 2.3. Concerning the assumptions of Theorem 2.2 let us note:

(a) The assumption that P_n and $A/z^{a-\partial\mathcal{A}}$ have at most a simple zero in common is no loss of generality. This can always be obtained by reducing common zeros of P_n and A (note that with z_j, $|z_j| \neq 1$, also $1/\overline{z_j}$ has to be reduced in A).

(b) The conditions (2.8) and (2.9) resp. condition (2.11) are necessary to fix the sign of Q_{n+l-2v} resp. $(i\Omega_n A - P_n B)/z^{a+l/2-w}$ at the zeros of A and at $z=0$, because these polynomials appear in quadratic form in (2.7) resp. (2.10). The opposite sign in (2.8) and (2.9) resp. in (2.11) would destroy the orthogonality property of P_n.

Note that the order of the O-term in (2.11) does not depend on the degree n. Using similar methods as in the proof of Theorem 2.2 (compare (4.9)–(4.11)) we see that condition (2.11) follows from (2.10) if for example $a + l/2 - w = 0$, $n > 0$ and $P_n(0) \neq 0$.

(c) If a polynomial P_{n_0} of degree n_0, $n_0 \in \mathbb{N}_0$, fulfills an infinite "lower" orthogonality property with respect to $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$, i.e., $\mathcal{L}(z^\tau P_{n_0}; \mathcal{A}, \mathcal{W}, \lambda) = 0$ for all $\tau \in \mathbb{N}$ (this case is not contained in Theorem 2.2), then

$$P_{n_0+k}(z) := z^k P_{n_0}(z), \qquad k \in \mathbb{N}_0, \tag{2.12}$$

are orthogonal polynomials with respect to $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$ (compare Theorem 3.8(a) below), and the polynomials R, W, A and B can be given explicitly as

$$R(z) = (1-z)^2 \qquad (\text{i.e. } E_l = E_1 = [0, 2\pi]), \qquad W(z) = i(1-z)$$

$$A(z) = \frac{i}{b} P^*_{n_0}(z)\, P_{n_0}(z), \qquad B(z; W, \lambda) = \frac{i}{2b} (\Omega^*_{n_0}(z)\, P_{n_0}(z) - P^*_{n_0}(z)\, \Omega_{n_0}(z)),$$

where $b \in \mathbb{R} \backslash \{0\}$ is given by $P_{n_0}(z)\, \Omega^*_{n_0}(z) + P^*_{n_0}(z)\, \Omega_{n_0}(z) = 2bz^{n_0}$. In this case we have by Theorem 2.1

$$\mathcal{L}\left(\frac{x+z}{x-z}; \mathcal{A}, \mathcal{W}, \lambda\right) = \frac{\Omega^*_{n_0}(z)}{P^*_{n_0}(z)}.$$

This means that the polynomials defined in (2.12) are exactly the Bernstein–Szegő polynomials, mentioned in Section 1, if $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$ is positive definite.

Next we show that the polynomial Q_{n+l-2v} from Theorem 2.2 fulfills an orthogonality property with respect to $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{V}, \lambda)$ if $f(\varphi; \mathcal{A}, \mathcal{V})$ is integrable, which for instance is satisfied for each choice of $\mathcal{W}$ if $\mathcal{R}$ has only simple zeros.

COROLLARY 2.4. *Let the polynomials* $R, V, W, A, B := B(\cdot; A, W, \lambda)$ *be given as in Theorem 2.2. For an integer* $n \geqslant \max\{\partial\mathscr{A} + l/2 - w, 2v - l\}$ *let* $P_n \in \mathbb{P}_n^{\mathbb{C}}, \partial P_n = n$, *such that* $\mathscr{L}(z^{-j}P_n; \mathscr{A}, \mathscr{W}, \lambda) = 0$ *for all* $j \in (0, ..., n + \mu - 1]$, $\mu \in \mathbb{N}_0$. *If* $f(\varphi; \mathscr{A}, \mathscr{V})$ *is integrable, then the polynomial*

$$Q_{n+l-2v}(z) := \frac{i\Omega_n(z)\, A(z) - P_n(z)\, B(z)}{z^{a+l/2-w}} \in \mathbb{P}_{n+l-2v}^{\mathbb{C}} \tag{2.13}$$

is orthogonal with respect to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$, *i.e.*,

$$\mathscr{L}(z^{-j}Q_{n+l-2v}; \mathscr{A}, \mathscr{V}, \lambda) = 0 \qquad \textit{for} \quad j \in (0, ..., (n+l-2v) + \mu - 1]. \tag{2.14}$$

Finally, let us give an integral representation of Q_{n+l-2v} and P_n in terms of P_n and Q_{n+l-2v}, respectively.

THEOREM 2.5. *Let the polynomial* P_n *of degree* n *be orthogonal with respect to* $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ *and let* Q_{n+l-2v} *be given as in* (2.13). *Further suppose that* R *has only simple zeros.*

(a) *For* $n \geqslant \max\{\partial\mathscr{A} + l/2 - w, 2v - l, 1\}$ *one has*

l even:

$$\begin{aligned} Q_{n+l-2v}(z) = iz^{w-l/2} \Bigg[& \frac{1}{2\pi} \int_{E_l} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \\ & \times (P_n(e^{i\varphi})\, e^{-iw\varphi} W(e^{i\varphi}) - P_n(z)\, z^{-w} W(z)) \frac{1}{r(\varphi)}\, d\varphi \\ & - \frac{1}{2\pi} \int_{E_l} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} (e^{-ia\varphi} A(e^{i\varphi}) - z^{-a} A(z)) \\ & \times P_n(e^{i\varphi})\, f(\varphi; \mathscr{A}, \mathscr{W})\, d\varphi \\ & + z^{-a} A(z)\, \mathscr{G}\left(\frac{x+z}{x-z} P_n; \mathscr{A}, \mathscr{W}, \lambda \right) \Bigg]; \end{aligned} \tag{2.15}$$

l odd:

$$\begin{aligned} Q_{n+l-2v}(z) = iz^{w-(l-1)/2} \Bigg[& \frac{1}{\pi} \int_{E_l} \frac{e^{i(\varphi/2)}}{e^{i\varphi} - z} \\ & \times (P_n(e^{i\varphi})\, e^{-iw\varphi} W(e^{i\varphi}) - P_n(z)\, z^{-w} W(z)) \frac{1}{r(\varphi)}\, d\varphi \end{aligned}$$

$$-\frac{1}{2\pi}\int_{E_l}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\left(\frac{2}{e^{i\varphi}+z}\,e^{-i(a-1/2)\varphi}A(e^{i\varphi})-z^{-(a+1/2)}A(z)\right)$$

$$\times P_n(e^{i\varphi})\,f(\varphi;\mathscr{A},\mathscr{W})\,d\varphi$$

$$+z^{-(a+1/2)}A(z)\,\mathscr{G}\left(\frac{x+z}{x-z}\,P_n;\mathscr{A},\mathscr{W},\lambda\right)\Bigg].\tag{2.16}$$

(b) *If* $f(\varphi;\mathscr{A},\mathscr{V})$ *is integrable and* $n\geqslant\max\{\partial\mathscr{A}+l/2-w,\,2v-l+1\}$ *then*

l even:

$$P_n(z)=iz^{v-l/2}\Bigg[\frac{1}{2\pi}\int_{E_l}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}$$

$$\times(Q_{n+l-2v}(e^{i\varphi})\,e^{-iv\varphi}V(e^{i\varphi})-Q_{n+l-2v}(z)\,z^{-v}V(z))\,\frac{1}{r(\varphi)}\,d\varphi$$

$$-\frac{1}{2\pi}\int_{E_l}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,(e^{-ia\varphi}A(e^{i\varphi})-z^{-a}A(z))$$

$$\times Q_{n+l-2v}(e^{i\varphi})\,f(\varphi;\mathscr{A},\mathscr{V})\,d\varphi$$

$$+z^{-a}A(z)\,\mathscr{G}\left(\frac{x+z}{x-z}\,Q_{n+l-2v};\mathscr{A},\mathscr{V},\lambda\right)\Bigg];\tag{2.17}$$

l odd:

$$P_n(z)=iz^{v-(l-1)/2}\Bigg[\frac{1}{\pi}\int_{E_l}\frac{e^{i(\varphi/2)}}{e^{i\varphi}-z}$$

$$\times(Q_{n+l-2v}(e^{i\varphi})\,e^{-iv\varphi}V(e^{i\varphi})-Q_{n+l-2v}(z)\,z^{-v}V(z))\,\frac{1}{r(\varphi)}\,d\varphi$$

$$-\frac{1}{2\pi}\int_{E_l}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\left(\frac{2}{e^{i\varphi}+z}\,e^{-i(a-1/2)\varphi}A(e^{i\varphi})-z^{-(a+1/2)}A(z)\right)$$

$$\times Q_{n+l-2v}(e^{i\varphi})\,f(\varphi;\mathscr{A},\mathscr{V})\,d\varphi$$

$$+z^{-(a+1/2)}A(z)\,\mathscr{G}\left(\frac{x+z}{x-z}\,Q_{n+l-2v};\mathscr{A},\mathscr{V},\lambda\right)\Bigg].\tag{2.18}$$

3. Preliminary Results and Proof of Theorem 2.1

In order to simplify the notation we assume that $E_l \subseteq [0, 2\pi]$. This is no essential loss of generality since all following arguments remain valid for a set $E_l \subseteq [a, a+2\pi]$, $a \in \mathbb{R}$, which can be seen by a simple shift (or a rotation on the unit circle). To prove our main results let us first study the expression

$$\lim_{\varrho \to 1^-} \sqrt{R(\varrho e^{i\varphi})}, \qquad R \text{ from (2.2)},$$

in some detail. Some preliminaries: we interprete $\sqrt{z}$ as a function in z on a two-sheeted Riemann surface $\mathscr{F}$ having a cut along the negative real axis, i.e., on $\mathscr{F}$ we distinguish between two complex numbers, the arguments of which differ by exactly 2π, but we identify two complex numbers, whose arguments differ by exactly 4π. Then the function

$$\sqrt{\cdot}: \mathscr{F} \to \mathbb{C}, \qquad z \mapsto \sqrt{z} := \sqrt{\varrho}\, e^{i(\alpha/2)}, \qquad z = \varrho e^{i\alpha}, \tag{3.1}$$

where $\varrho \in \mathbb{R}_0^+$ and $\alpha \in [0, 4\pi)$, is uniquely defined and analytic on $\mathscr{F} \backslash \{0\}$. The polynomial R induces the function (of the same name)

$$R(\cdot): \mathbb{C} \to \mathscr{F}, \qquad z \mapsto R(z). \tag{3.2}$$

Combining (3.1) and (3.2) we get that

$$\sqrt{R(\cdot)}: \mathbb{C} \backslash Int(\Gamma_{E_l}) \to \mathbb{C}, \qquad z \mapsto \sqrt{R(z)} \tag{3.3}$$

is uniquely defined and analytic on $\mathbb{C} \backslash \Gamma_{E_l}$, which will be shown in Lemma 3.1 and Remark 3.2 below. Further we set

$$\sqrt{R(e^{i\varphi})} := \lim_{\varrho \to 1^-} \sqrt{R(\varrho e^{i\varphi})} = \sqrt{\lim_{\varrho \to 1^-} R(\varrho e^{i\varphi})}, \qquad \varphi \in E_l. \tag{3.4}$$

It will also turn out (see Remark 3.2) that we have $\lim_{\varrho \to 1^-} \sqrt{R(\varrho e^{i\varphi})} = \lim_{\varrho \to 1^+} \sqrt{R(\varrho e^{i\varphi})}$ for $\varphi \in [0, 2\pi] \backslash E_l$, but $\lim_{\varrho \to 1^-} \sqrt{R(\varrho e^{i\varphi})} = -\lim_{\varrho \to 1^+} \sqrt{R(\varrho e^{i\varphi})}$ for $\varphi \in Int(E_l)$.

Lemma 3.1. *For given points* $0 =: \varphi_0 \leqslant \cdots \leqslant \varphi_{2l} \leqslant \varphi_{2l+1} := 2\pi$, $l \in \mathbb{N}$, *let* $R \in \mathbb{P}_{2l}^{\mathbb{C}}$ *be given as in* (2.2). *Then for that branch of* $\sqrt{R(e^{i\varphi})}$ *defined in* (3.4), *called henceforth positive branch of* $\sqrt{R(e^{i\varphi})}$, *one has*

$$\sqrt{R(e^{i\varphi})} = \begin{cases} (-1)^j\, i e^{i(l/2)\varphi} \sqrt{|R(e^{i\varphi})|}, & \varphi \in [\varphi_{2j-1}, \varphi_{2j}]\ j = 1, ..., l \\ (-1)^j\, e^{i(l/2)\varphi} \sqrt{|R(e^{i\varphi})|}, & \varphi \in [\varphi_{2j}, \varphi_{2j+1}]\ j = 0, ..., l. \end{cases} \tag{3.5}$$

Proof. We first show the assertion for the case that R has simple zeros only, i.e., $0<\varphi_1< \cdots <\varphi_{2l}$. In the following $\arg(z)$ denotes the argument of a complex number z and is understood as a continuous function from $\mathbb{C}$ to $\mathbb{R}$ (and not necessarily to $[0, 2\pi)$ or $[0, 4\pi)$); i.e., each time when z circles around the origin in the positive sense, $\arg(z)$ increases by 2π.

In view of (3.5) and (3.1) we have to show that

$$\arg(R(e^{i\varphi}))=l\varphi-j\pi \qquad \text{for} \quad \varphi\in(\varphi_j, \varphi_{j+1}), \qquad j=0, ..., 2l$$

or equivalently that

$$\arg(e^{-il\varphi}R(e^{i\varphi}))=-j\pi \qquad \text{for} \quad \varphi\in(\varphi_j, \varphi_{j+1}), \qquad j=0, ..., 2l, \quad (3.6)$$

where we assume that $\arg(e^{-il\varphi}R(e^{i\varphi}))=0$ for $\varphi\in(0, \varphi_1)$ (note that $e^{-il\varphi}R(e^{i\varphi})$ is a real trigonometric polynomial).

To prove (3.6) we need an explicit representation of $z^{-l}R(z)$ for $z=\varrho e^{i\varphi}$, $\varrho\in(0, 1]$, $\varphi\in[0, 2\pi]$. Since the polynomial R is selfreciprocal it can be written in the form

$$R(z)=\sum_{k=0}^{2l}(a_k+ib_k)\,z^k,$$
$$a_k=a_{2l-k}\in\mathbb{R}, \quad b_k=-b_{2l-k}\in\mathbb{R}, \quad |a_0|+|b_0|\neq 0.$$

Thus we obtain

$$\operatorname{Re}(z^{-l}R(z)|_{z=\varrho e^{i\varphi}})=a_l+\sum_{k=l+1}^{2l}\left(\varrho^{k-l}+\frac{1}{\varrho^{k-l}}\right)$$
$$\times(a_k\cos(k-l)\,\varphi-b_k\sin(k-l)\,\varphi)$$
$$\operatorname{Im}(z^{-l}R(z)|_{z=\varrho e^{i\varphi}})=\sum_{k=l+1}^{2l}\left(\varrho^{k-l}-\frac{1}{\varrho^{k-l}}\right)$$
$$\times(b_k\cos(k-l)\,\varphi+a_k\sin(k-l)\,\varphi)$$

and as a consequence

$$\mathscr{R}(\varphi):=\operatorname{Re}(z^{-l}R(z)|_{z=\varrho e^{i\varphi}})|_{\varrho=1}$$
$$=a_l+2\sum_{k=l+1}^{2l}(a_k\cos(k-l)\,\varphi-b_k\sin(k-l)\,\varphi)$$
$$\mathscr{R}_1(\varphi):=\frac{d}{d\varrho}\operatorname{Im}(z^{-l}R(z)|_{z=\varrho e^{i\varphi}})|_{\varrho=1}$$
$$=2\sum_{k=l+1}^{2l}(k-l)(b_k\cos(k-l)\,\varphi+a_k\sin(k-l)\,\varphi),$$

i.e.,

$$\mathscr{R}'(\varphi) = -\mathscr{R}_1(\varphi).$$

Since all zeros of $\mathscr{R}(\varphi)$ are simple, and $\mathscr{R}(\varphi) > 0$ on $(0, \varphi_1)$, we have

$$\operatorname{sgn} \mathscr{R}_1(\varphi_j) = -\operatorname{sgn} \mathscr{R}'(\varphi_j) = (-1)^{j+1}, \qquad j = 1, ..., 2l.$$

Thus by the definition of $\mathscr{R}_1$ and by the fact that $\operatorname{Im}(e^{-il\varphi} R(e^{i\varphi})) = 0$ on $[0, 2\pi]$ one has for all ϱ sufficiently near to 1^-

$$\operatorname{Im}(z^{-l} R(z)|_{z = \varrho e^{i\varphi_j}}) \begin{cases} < 0, & j \in \{1, 3, ..., 2l-1\} \\ > 0, & j \in \{2, 4, ..., 2l\}. \end{cases} \tag{3.7}$$

Because of the continuity of $\operatorname{Im}(z^{-l} R(z))$ in a neighbourhood of the unit circumference $|z| = 1$ property (3.7) also holds on $[\varphi_j - \varepsilon_j, \varphi_j + \varepsilon_j]$, $j = 1, ..., 2l$ for $\varepsilon_j > 0$ sufficiently small.

Now let γ be a closed, positively oriented curve in the complex plane, starting at the point $z = 1$, which coincides with the unit circumference except for the regions $\{e^{i\varphi} : \varphi \in [\varphi_j - \varepsilon_j, \varphi_j + \varepsilon_j], \varepsilon_j > 0, j = 1, ..., 2l\}$, where γ lies in the interior of the unit disk as shown in Fig. 2.1. We choose the values ε_j, $j = 1, ..., 2l$, sufficiently small such that condition (3.7) holds on γ, when γ lies in the interior of the unit disk.

For $\varphi \in [0, \varphi_1 - \varepsilon_1]$, where $\varphi_1 - \varepsilon_1 > 0$, we have $\mathscr{R}(\varphi) = e^{-il\varphi} R(e^{i\varphi}) > 0$ and $\arg(e^{-il\varphi} R(e^{i\varphi})) = 0$. Because of (3.7) one has $\operatorname{Im}(z^{-l} R(z)) < 0$ on γ near $e^{i\varphi_1}$ and γ in the interior of the unit disk, and further $e^{-il\varphi} R(e^{i\varphi}) < 0$,

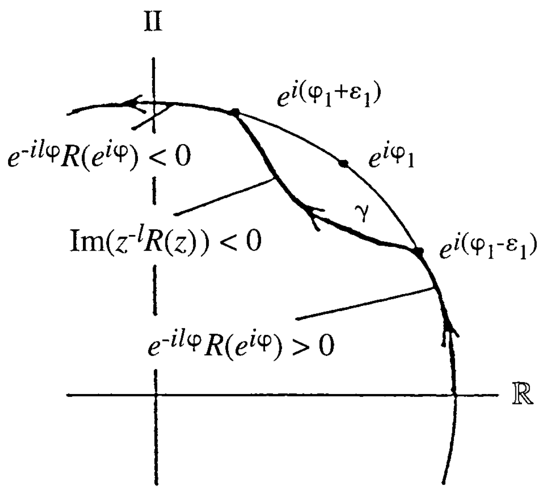

FIGURE 2.1

$\varphi \in [\varphi_1 + \varepsilon_1, \varphi_2 - \varepsilon_2]$, where $\varphi_1 + \varepsilon_1 < \varphi_2 - \varepsilon_2$. Thus we see that $z^{-l}R(z)$ describes half a circle around the origin in negative sense while z passes the point $z^{i\varphi_1}$ on the curve γ, i.e., $\arg(z^{-l}R(z))$ decreases by π.

The same considerations for the remaining points $e^{i\varphi_j}$, $j=2, ..., 2l$, lead to the desired assertion (3.6), which proves the lemma if R has simple zeros only.

If R has multiple roots then this can be considered as a limit case when simple zeros coincide and the assertion follows. ∎

Remark 3.2. Let us state some comments concerning (2.4): if one examines the methods in the proof of Lemma 3.1 in detail, one sees that the radial boundary values $\lim_{\varrho \to 1^-} \arg(R(\varrho e^{i\varphi})) =: z_1$ and $\lim_{\varrho \to 1^+} \arg(R(\varrho e^{i\varphi})) =: z_2$ differ by exactly 2π if $\varphi \in Int(E_l)$, i.e., $z_1 \neq z_2$ on the Riemann surface $\mathscr{F}$. Thus, by (3.3) and (3.1) we have $\sqrt{R(z_1)} \neq \sqrt{R(z_2)}$, although $z_1 = z_2$ on the complex plane $\mathbb{C}$. If $\varphi \in [0, 2\pi] \backslash E_l$ we have (compare again the proof of Lemma 3.1)

$$\lim_{\varrho \to 1^-} \arg(R(\varrho e^{i\varphi})) = \lim_{\varrho \to 1^+} \arg(R(\varrho e^{i\varphi})) + \varepsilon \cdot 4\pi, \qquad \varepsilon \in \{-1, 1\},$$

i.e., $\lim_{\varrho \to 1^-} R(\varrho e^{i\varphi}) = \lim_{\varrho \to 1^+} R(\varrho e^{i\varphi})$ on $\mathbb{C}$ *and* on $\mathscr{F}$, and thus (compare (3.1))

$$\lim_{\varrho \to 1^-} \sqrt{R(\varrho e^{i\varphi})} = \lim_{\varrho \to 1^+} \sqrt{R(\varrho e^{i\varphi})} \qquad \text{for} \quad \varphi \in [0, 2\pi] \backslash E_l.$$

The following lemma is needed to state and to prove Theorem 2.1.

LEMMA 3.3. *Let the polynomials R, V, W, A be given as in* (2.2) *and let* $\lambda = (\lambda_1, ..., \lambda_{m^*}) \in \Delta_{m^*}$. *Then there exists a uniquely determined polynomial* $B := B(\cdot\,; A, W, \lambda) \in \mathbb{P}_{2a}^{\mathbb{C}}$, *which satisfies the following conditions*

$$B = B_{2a}^{(*)}$$

$$(VB)^{(\nu)}(z_j) = -\lambda_j (z^{a+l/2-w} \sqrt{R})^{(\nu)}(z_j), \qquad \nu = 0, ..., m_j - 1, \quad j = 1, ..., m^*$$

$$\lim_{z \to 0} \frac{B(z) + z^{a+l/2-w} \dfrac{W(z)}{\sqrt{R(z)}}}{iA(z)} \in \mathbb{R}. \tag{3.8}$$

Proof. We denote the multiplicity of the zero of A at $z=0$ by $m := a - \partial\mathscr{A}$ and set

$$\tilde{B}(z) := C(z) + z^m D(z) + C_{2a}^{(*)}(z) \in \mathbb{P}_{2a}^{\mathbb{C}},$$

where the polynomials $C \in \mathbb{P}^{\mathbb{C}}_{m-1}$ and $D \in \mathbb{P}^{\mathbb{C}}_{2(a-m)}$ are chosen in such a way that the system

$$\tilde{B}^{(v)}(z_j) = -\lambda_j \left(z^{a+l/2-w} \frac{W}{\sqrt{R}} \right)^{(v)} (z_j),$$

$$v = 0, ..., m_j - 1, \quad j = 1, ..., m^* \tag{3.9}$$

$$\tilde{B}(z) + z^{a+l/2-w} \frac{W(z)}{\sqrt{R(z)}} = O(z^m), \qquad \text{as} \quad z \to 0$$

is fulfilled. Note that $z_j \notin \Gamma_{E_l}$ and that $\sqrt{R}$ is analytic on $\mathbb{C} \backslash \Gamma_{E_l}$, thus the derivatives in (3.9) exist and further $a + l/2 - w \in \mathbb{N}_0$ by Assumption 1.1(b) and (2.1).

The polynomial C is uniquely determined by the second equation in (3.9) and D uniquely up to a multiplicative constant by the first equations. Because $z = 0$ is not contained in $\{z_1, ..., z_{m^*}\}$ we obtain uniqueness of D if we fix $D(0)$.

Since $1/\overline{z_j}$ is a zero of A/z^m of the same multiplicity as z_j, the polynomial $\tilde{B}^{(*)}_{2a}$ solves the system (3.9), too (recall $R = R^*$, $V = V^*$, $\lambda_{j_1} = \lambda_{j_2}$ if $z_{j_1} = 1/\overline{z_{j_2}}$), and hence the polynomial $\frac{1}{2}(\tilde{B}(z) + \tilde{B}^{(*)}_{2a}(z))$, which is uniquely determined if we fix $D(0)$. Now we will choose $D(0)$ such that the polynomial

$$B(z) := \tfrac{1}{2}(\tilde{B}(z) + \tilde{B}^{(*)}_{2a}(z))$$

satisfies (3.8). Since B solves (3.9), only the last condition in (3.8) remains to be shown. From (3.9) we obtain

$$\tilde{B}(z) - \tilde{B}^{(*)}_{2a}(z) = \alpha i A(z), \qquad \alpha \in \mathbb{R} \text{ depending on } D(0), \tag{3.10}$$

where the fact that $\alpha \in \mathbb{R}$ can be seen from

$$\bar{\alpha} i A(z) = -(\alpha i A(z))^{(*)}_{2a} = -(\tilde{B}(z) - \tilde{B}^{(*)}_{2a}(z))^{(*)}_{2a} = \tilde{B}(z) - \tilde{B}^{(*)}_{2a}(z) = \alpha i A(z).$$

If we write B in the form $B(z) = C(z) + C^{(*)}_{2a}(z) + \frac{1}{2} z^m (D(z) + D^{(*)}_{2(a-m)}(z))$, we obtain

$$\lim_{z \to 0} \frac{B(z) + z^{a+l/2-w} \dfrac{W(z)}{\sqrt{R(z)}}}{iA(z)} = \frac{\dfrac{1}{2}(d_0 + \bar{d}_{2(a-m)}) + r_m}{a_0},$$

where $d_0 = D(0)$, $\bar{d}_{2(a-m)} = D^{(*)}_{2(a-m)}(0)$ (i.e., the leading coefficient of $\bar{D}(z)$), $a_0 = (iA/z^m)|_{z=0} \neq 0$ and r_m is given by the expansion

$$z^{a+l/2-w} \frac{W(z)}{\sqrt{R(z)}} = \sum_{j=0}^{\infty} r_j z^j.$$

From (3.10) we get $\bar{d}_{2(a-m)} = d_0 - \alpha a_0$. Hence we can choose the free parameter d_0 such that $\operatorname{Im} d_0/a_0 = -\operatorname{Im} r_m/a_0$ (note that a_0 and r_m are independent of d_0) and get

$$\lim_{z \to 0} \frac{B(z) + z^{a+l/2-w} \dfrac{W(z)}{\sqrt{R(z)}}}{iA(z)} = \frac{d_0 + r_m}{a_0} - \frac{\alpha}{2} \in \mathbb{R}.$$

In Remark 3.5 we will get the uniqueness of B as a by-product of Theorem 2.1. ▮

In Corollary 3.6 below (after the proof of Theorem 2.1) we will give an explicit representation of the polynomial $B := B(\cdot; A, W, \lambda)$, which will be needed to prove Theorem 2.5.

We now define the function

$$F(z; A, W, \lambda) := -\frac{B(z) + z^{a+l/2-w} \dfrac{W(z)}{\sqrt{R(z)}}}{iA(z)}, \tag{3.11}$$

which is analytic on $\mathbb{C} \setminus (\Gamma_{E_l} \cup \{z_j : \lambda_j = -1\})$ by (3.8) and assumption 1.1(b) (note that $a + l/2 - w \in \mathbb{N}_0$). At the points z_j, $\lambda_j = -1$, the function $F(z; A, W, \lambda)$ has poles of order m_j. On the set Γ_{E_l} we define

$$F(e^{i\varphi}; A, W, \lambda) := \lim_{\varrho \to 1^-} F(\varrho e^{i\varphi}; A, W, \lambda), \qquad \varphi \in E_l. \tag{3.12}$$

For $\lambda^0 := (1, 1, ..., 1) \in \Delta_{m^*}$ we abbreviate

$$F(z; A, W) := F(z; A, W, \lambda^0) \qquad \text{and} \qquad B(z; A, W) := B(z; A, W, \lambda^0). \tag{3.13}$$

The function $F(z; A, W)$ is analytic on $\mathbb{C} \setminus \Gamma_{E_l}$ and plays a crucial role in developping our theory, as the following lemma shows.

LEMMA 3.4. *Let the polynomials R, W, A and the function $f(\varphi; \mathscr{A}, \mathscr{W})$ be given and let $F(z; A, W)$ be defined as above. Then we have*

$$\operatorname{Re} F(e^{i\varphi}; A, W) = f(\varphi; \mathscr{A}, \mathscr{W}) \tag{3.14}$$

and

$$F(z; A, W) = \frac{1}{2\pi} \int_{E_l} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} f(\varphi; \mathscr{A}, \mathscr{W})\, d\varphi, \qquad z \in \mathbb{C} \backslash \Gamma_{E_l}. \tag{3.15}$$

Proof. Let $\sqrt{R(e^{i\varphi})}$ be defined as in (2.4) resp. (3.4), then we have by (3.11)

$$\begin{aligned} &\operatorname{Re} F(e^{i\varphi}; A, W) \\ &= \frac{\operatorname{Re}\left(iB(e^{i\varphi}; A, W)\, \overline{A(e^{i\varphi})} + ie^{i(a + l/2 - w)\varphi} \dfrac{W(e^{i\varphi})}{\sqrt{R(e^{i\varphi})}}\, \overline{A(e^{i\varphi})}\right)}{|A(e^{i\varphi})|^2} \\ &= \frac{\operatorname{Re}\left(ie^{i(a + l/2 - w)\varphi} \dfrac{W(e^{i\varphi})}{\sqrt{R(e^{i\varphi})}}\, \overline{A(e^{i\varphi})}\right)}{|A(e^{i\varphi})|^2}, \end{aligned}$$

since $\operatorname{Re}(iB(e^{i\varphi}; A, W)\, \overline{A(e^{i\varphi})}) = 0$ by $B = B_{2a}^{(*)}$ and $A = A_{2a}^{(*)}$. Furthermore we obtain from (3.5) that

$$\begin{aligned} &ie^{i(a + l/2 - w)\varphi} \frac{W(e^{i\varphi})}{\sqrt{R(e^{i\varphi})}}\, \overline{A(e^{i\varphi})} \\ &= \begin{cases} (-1)^j \dfrac{\mathscr{W}(\varphi)\, \mathscr{A}(\varphi)}{\sqrt{|\mathscr{R}(\varphi)|}} \in \mathbb{R}, & \varphi \in (\varphi_{2j-1}, \varphi_{2j}), \qquad j = 1, ..., l \\ (-1)^j\, i\, \dfrac{\mathscr{W}(\varphi)\, \mathscr{A}(\varphi)}{\sqrt{|\mathscr{R}(\varphi)|}} \in i\mathbb{R}, & \varphi \in (\varphi_{2j}, \varphi_{2j+1}), \qquad j = 0, ..., l. \end{cases} \end{aligned}$$

Taking into consideration the fact that $|A(e^{i\varphi})|^2 = \mathscr{A}^2(\varphi)$ the assertion (3.14) follows.

By assumption 1.1(a) we have $f(\varphi; \mathscr{A}, \mathscr{W}) \in L^q[0, 2\pi]$, $q \in [1, 2)$, and by the third property in (3.8) one has $F(0; A, W) \in \mathbb{R}$. Thus (3.15) can be derived from (3.14) and [17, Ch.I.D and V.B] for $|z| < 1$. Since both functions $F(z; A, W)$ and $1/2\pi \int_{E_l} ((e^{i\varphi} + z)/(e^{i\varphi} - z))\, f(\varphi; \mathscr{A}, \mathscr{W})\, d\varphi$ are analytic on $\mathbb{C} \backslash \Gamma_{E_l}$ and coincide on $|z| < 1$, representation (3.15) holds by the Identity-Theorem for analytic functions on the whole region $\mathbb{C} \backslash \Gamma_{E_l}$. ∎

Before we prove Theorem 2.1 let us note the simple fact that a moment generating linear functional $\mathscr{L}$, not necessarily of the form (1.17), whose moments $c_j := \mathscr{L}(z^{-j})$, $j \in \mathbb{Z}$, satisfy

$$c_{-j} = \bar{c}_j, \qquad j \in \mathbb{N}_0 \qquad \text{and} \qquad \frac{1}{\limsup_{j \to \infty} \sqrt[j]{|c_j|}} = \varrho, \qquad \varrho > 0, \tag{3.16}$$

induces in a natural way the following analytic function

$$F(z) := \mathscr{L}\left(\frac{x+z}{x-z}\right) = c_0 + 2 \sum_{j=1}^{\infty} c_j z^j, \qquad |z| < \varrho, \qquad \text{where } \mathscr{L} \text{ acts on } x. \tag{3.17}$$

We now are able to proof Theorem 2.1.

Proof of Theorem 2.1. By definition (3.11) we have

$$F(z; A, W, \lambda) = F(z; A, W) - \frac{C(z)}{iA(z)},$$

$$\text{where} \quad C(z) := B(z; A, W, \lambda) - B(z; A, W). \tag{3.18}$$

From $C = C_{2a}^{(*)}$, $A = A_{2a}^{(*)}$ and the third property in (3.8) one gets $\partial C \leqslant \partial A$, thus we can write

$$\frac{C(z)}{iA(z)} = K_0 + \frac{D(z)}{iA(z)}, \qquad D \in \mathbb{P}_{\partial A - 1}^{\mathbb{C}}$$

where $D^{(v)}(z_j) = C^{(v)}(z_j)$ for $j = 1, ..., m^*$, $v = 0, ..., m_j - 1$, $D^{(v)}(0) = C^{(v)}(0)$ for $v = 0, ..., a - \partial\mathscr{A} - 1$, and by (3.18)

$$K_0 = -\overline{\left[\left.\frac{C(z)}{iA(z)}\right|_{z=0}\right]} = F(0; A, W, \lambda) - F(0; A, W) \in \mathbb{R}$$

$$(\text{note } C = C_{2a}^{(*)} \text{ and } A = A_{2a}^{(*)}). \tag{3.19}$$

Let us denote $z_0 := 0$, $m_0 := a - \partial\mathscr{A}$ (i.e., m_0 is the order of the zero of A at $z = 0$) and $\lambda_0 := 1$ then a partial fraction expansion gives

$$\frac{C(z)}{iA(z)} = K_0 + \sum_{j=1}^{m^*} \sum_{v=0}^{m_j - 1} \frac{\tilde{\mu}_{j,v}}{(z - z_j)^{v+1}}, \tag{3.20}$$

with

$$\tilde{\mu}_{j,v} = \frac{\left(\frac{C}{iA_j}\right)^{(m_j - 1 - v)}(z_j)}{(m_j - 1 - v)!}$$

$$= \frac{(1 - \lambda_j)\left(\frac{z^{a + l/2 - w} W}{iA_j \sqrt{R}}\right)^{(m_j - 1 - v)}(z_j)}{(m_j - 1 - v)!}, \qquad A_j(z) := \frac{A(z)}{(z - z_j)^{m_j}},$$

where the second equality follows with the help of the interpolation property in (3.8). Let now $\varrho := \min\{|z_j| : \lambda_j = -1\}$, then by definition $\varrho > 0$. By applying the Cauchy-product $(\nu+1)$-times, $\nu \in \mathbb{N}_0$, or by using the identity

$$\frac{1}{(z-z_j)^{\nu+1}} = \frac{1}{\nu!}\frac{d^\nu}{dx^\nu}\left(-\frac{1}{2z}\left(\frac{x+z}{x-z}-1\right)\right)(z_j)$$
$$= -\frac{1}{\nu!}\frac{d^\nu}{dx^\nu}\left(\sum_{k=0}^{\infty}\frac{z^k}{x^{k+1}}\right)(z_j)$$

one obtains for $|z| < |z_j|$

$$\frac{1}{(z-z_j)^{\nu+1}} = \frac{(-1)^{\nu+1}}{\nu!}\frac{1}{z_j^{\nu+1}}\sum_{k=0}^{\infty}(k+1)(k+2)\cdots(k+\nu)\left(\frac{z}{z_j}\right)^k$$
$$= -\frac{1}{\nu!}\sum_{k=0}^{\infty}\left(\frac{1}{x^{k+1}}\right)^{(\nu)}(z_j)\cdot z^k$$

and thus for $|z| < \varrho$ by (3.20) (note $\tilde{\mu}_{0,\nu} = 0$)

$$\frac{C(z)}{iA(z)} = K_0 - \sum_{j=1}^{m^*}\sum_{\nu=0}^{m_j-1}\tilde{\mu}_{j,\nu}\frac{1}{\nu!}\left(\frac{1}{x}\right)^{(\nu)}(z_j)$$
$$-2\sum_{k=1}^{\infty}\frac{1}{2}\sum_{j=1}^{m^*}\sum_{\nu=0}^{m_j-1}\tilde{\mu}_{j,\nu}\frac{1}{\nu!}\left(\frac{1}{x^{k+1}}\right)^{(\nu)}(z_j)\cdot z^k. \tag{3.21}$$

From (3.19) and (3.21) we get

$$-K_0 = K_0 - \sum_{j=1}^{m^*}\sum_{\nu=0}^{m_j-1}\tilde{\mu}_{j,\nu}\frac{1}{\nu!}\left(\frac{1}{x}\right)^{(\nu)}(z_j)$$

and thus

$$K_0 = \frac{1}{2}\sum_{j=1}^{m^*}\sum_{\nu=0}^{m_j-1}\tilde{\mu}_{j,\nu}\frac{1}{\nu!}\left(\frac{1}{x}\right)^{(\nu)}(z_j).$$

Now by (3.15) (compare also (3.17)) we can write

$$F(z; A, W) = c_0 + 2\sum_{k=1}^{\infty}c_k z^k, \qquad |z| < 1,$$

where

$$c_k := \frac{1}{2\pi}\int_{E_l} e^{-ik\varphi} f(\varphi; \mathscr{A}, \mathscr{W})\, d\varphi.$$

Hence by (3.18) and (3.21), note that $F(\cdot; A, W, \lambda)$ is analytic on $|z|<\varrho$,

$$F(z; A, W, \lambda) = c_0^\lambda + 2\sum_{k=1}^{\infty} c_k^\lambda z^k, \qquad |z|<\varrho \tag{3.22}$$

with

$$c_k^\lambda = c_k + \frac{1}{2}\sum_{j=1}^{m^*}\sum_{\nu=0}^{m_j-1} \tilde{\mu}_{j,\nu}\frac{1}{\nu!}\left(\frac{1}{x^{k+1}}\right)^{(\nu)}(z_j) = \mathcal{L}(x^{-k}; \mathcal{A}, \mathcal{W}, \lambda),$$

where $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$ is given as in (1.17) with $\tilde{\mu}_{j,\nu} = (1-\lambda_j)\,\mu_{j,\nu}$ and $\mu_{j,\nu}$ from the theorem.

Now by the consideration (3.17) the assertion (2.5) follows for $|z|<\varrho$ and by the same arguments as at the end of the proof of Lemma 3.4 we obtain that (2.5) even holds on the larger set $\mathbb{C}\backslash(\Gamma_{E_l} \cup \{z_j : \lambda_j = -1\})$. ∎

Remark 3.5. Considering the second identity in (2.5) and noting that the left hand side of this equation does not depend on $B(\cdot; A, W, \lambda)$ we conclude that the polynomial $B(\cdot; A, W, \lambda)$ is uniquely determined by the system (3.8). This closes the gap in the proof of Lemma 3.3.

The following corollary gives an explicit representation of the polynomial $B(\cdot; A, W, \lambda)$.

COROLLARY 3.6. *Let the polynomials R, A, $B := B(\cdot; A, W, \lambda)$, the function $r(\varphi)$ and the functional $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$ be given and suppose that R has only simple zeros. Then the polynomial B has the following representation (where $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$ acts on x):*

l even:

$$\begin{aligned} &B(z; A, W, \lambda) \\ &\quad = iz^a\Bigg[\mathcal{L}\left(\frac{x+z}{x-z}(x^{-a}A(x) - z^{-a}A(z)); \mathcal{A}, \mathcal{W}, \lambda\right) \\ &\qquad - \frac{1}{2\pi}\int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}(e^{-iw\varphi}W(e^{i\varphi}) - z^{-w}W(z))\frac{1}{r(\varphi)}\,d\varphi\Bigg]. \end{aligned} \tag{3.23}$$

l odd and $w \in \mathbb{N}_0$:

$$\begin{aligned} &B(z; A, W, \lambda) \\ &\quad = iz^{a+1/2}\Bigg[\mathcal{L}\left(\frac{x+z}{x-z}\left(\frac{2}{x+z}x^{-(a-1/2)}A(x) - z^{-(a+1/2)}A(z)\right); \mathcal{A}, \mathcal{W}, \lambda\right) \\ &\qquad - \frac{1}{\pi}\int_{E_l} \frac{e^{i(\varphi/2)}}{e^{i\varphi}-z}(e^{-iw\varphi}W(e^{i\varphi}) - z^{-w}W(z))\frac{1}{r(\varphi)}\,d\varphi\Bigg]. \end{aligned} \tag{3.24}$$

l odd and $w \in \frac{1}{2}\mathbb{N}$:

$$\begin{aligned} & B(z; A, W, \lambda) \\ & = iz^a \Bigg[\mathscr{L}\left(\frac{x+z}{x-z}(x^{-a}A(x) - z^{-a}A(z)); \mathscr{A}, \mathscr{W}, \lambda\right) \\ & \quad - \frac{1}{\pi}\int_{E_l} \frac{e^{i(\varphi/2)}}{e^{i\varphi}-z}\left(\frac{e^{i\varphi}+z}{2}\, e^{-i(w+1/2)\varphi} W(e^{i\varphi}) - z^{-(w-1/2)} W(z)\right)\frac{1}{r(\varphi)}\, d\varphi\Bigg]. \end{aligned} \tag{3.25}$$

Proof. We first consider the case when l is even: From (3.11) and (2.5) we get

$$B(z; A, W, \lambda) = -iA(z)\, \mathscr{L}\left(\frac{x+z}{x-z}; \mathscr{A}, \mathscr{W}, \lambda\right) + iz^{a-w} W(z)\, \frac{iz^{l/2}}{\sqrt{R(z)}}. \tag{3.26}$$

From (3.5) we obtain

$$\operatorname{Re}\left(\frac{ie^{i(l/2)\varphi}}{\sqrt{R(e^{i\varphi})}}\right) = \begin{cases} \dfrac{1}{r(\varphi)} \in L_q(E_l), \quad q \in [1,2), & \varphi \in E_l \\ 0, & \varphi \notin E_l. \end{cases}$$

Since $iz^{l/2}/\sqrt{R(z)}$ is analytic on $|z|<1$ it can be derived from [17, Ch.I.D and Ch.V.B] (compare also the proof of Lemma 3.4)

$$\frac{iz^{l/2}}{\sqrt{R(z)}} = \frac{1}{2\pi}\int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, \frac{1}{r(\varphi)}\, d\varphi, \qquad |z|<1. \tag{3.27}$$

From (3.26) and (3.27) we get for $|z|<1$

$$\begin{aligned} B(z; A, W, \lambda) &= iz^a \Bigg[-z^{-a}A(z)\, \mathscr{L}\left(\frac{x+z}{x-z}; \mathscr{A}, \mathscr{W}, \lambda\right) \\ &\quad + z^{-w} W(z)\, \frac{1}{2\pi}\int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, \frac{1}{r(\varphi)}\, d\varphi\Bigg] \\ &= iz^a \Bigg[\mathscr{L}\left(\frac{x+z}{x-z}(x^{-a}A(x) - z^{-a}A(z)); \mathscr{A}, \mathscr{W}, \lambda\right) \\ &\quad - \frac{1}{2\pi}\int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, \mathscr{A}(\varphi)\, f(\varphi; \mathscr{A}, \mathscr{W})\, d\varphi \\ &\quad + \frac{1}{2\pi}\int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, z^{-w} W(z)\, \frac{1}{r(\varphi)}\, d\varphi\Bigg] \end{aligned}$$

$$B(z; A, W, \lambda) = iz^a \left[\mathcal{L}\left(\frac{x+z}{x-z}(x^{-a}A(x) - z^{-a}A(z)); \mathscr{A}, \mathscr{W}, \lambda\right) \right.$$
$$\left. -\frac{1}{2\pi}\int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}(\mathscr{W}(\varphi) - z^{-w}W(z))\frac{1}{r(\varphi)}\,d\varphi\right].$$

This is the assertion (3.23) for $|z|<1$.

We now show that the right hand side in (3.23) is as well a polynomial of degree $\leqslant 2a$, which gives the representation (3.23) on the whole complex plain. Let us restrict attention to the case $w \in \mathbb{N}_0$ (the methods for $w \in \frac{1}{2}\mathbb{N}$ are quite similar but calculation is more tedious). By Assumption 1.1(b) we get that $a \in \mathbb{N}_0$ and thus

$$z^a \mathcal{L}\left(\frac{x+z}{x-z}(x^{-a}A(x) - z^{-a}A(z)); \mathscr{A}, \mathscr{W}, \lambda\right) \in \mathbb{P}_{2a}^{\mathbb{C}} \qquad \text{(polynomial in } z\text{).} \tag{3.28}$$

Expanding the right hand side in (3.27) in a power series at $z=0$ we obtain that

$$\int_{E_l} e^{-ij\varphi}\frac{1}{r(\varphi)}\,d\varphi = 0 \qquad \text{for} \quad j \in \left[-\frac{l}{2}+1, ..., \frac{l}{2}-1\right]$$

and in a similar way as in (3.28) by using this orthogonality property we get

$$z^a \frac{1}{2\pi}\int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}(\mathscr{W}(\varphi) - z^{-w}W(z))\frac{1}{r(\varphi)}\,d\varphi \in \mathbb{P}_{2a}^{\mathbb{C}} \qquad \text{(polynomial in } z\text{)}$$

and thus (3.23) is proven.

The representations (3.24) and (3.25) can be obtained after some calculation by applying the methods we used to prove (3.23) to the polynomials $\tilde{R}(z) := R(z^2)$, $\tilde{A}(z) := A(z^2)$ and $\tilde{W}(z) := W(z^2)$. ∎

Remark 3.7. If $\lambda = \lambda^0 = (1, 1, ..., 1)$, i.e., no Dirac "mass-points" appear in $\mathcal{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$, the representations (3.23)–(3.25) take a more transparent form. For example in the case of (3.23) we have

$$B(z; A, W) = B(z; A, W, \lambda^0)$$
$$= iz^a \left[\frac{1}{2\pi}\int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}(e^{-ia\varphi}A(e^{i\varphi}) - z^{-a}A(z))\, f(\varphi; \mathscr{A}, \mathscr{W})\,d\varphi \right.$$
$$\left. -\frac{1}{2\pi}\int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}(e^{-iw\varphi}W(e^{i\varphi}) - z^{-w}W(z))\frac{1}{r(\varphi)}\,d\varphi\right].$$

At the end of this section we give a sufficient and necessary condition for a polynomial to be orthogonal with respect to the functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$, which will be needed for the proof of Theorem 2.2. In order to get these sufficient and necessary conditions we need the following general characterization theorem for orthogonal polynomials, which we have proven in [24].

THEOREM 3.8 (Peherstorfer and Steinbauer [24]). *Let a linear functional $\mathscr{L}$, which fulfills* (3.16), *be given and let F be the function defined by* (3.17). *Further let $P_n \in \mathbb{P}_n^{\mathbb{C}}$ of degree $\partial P_n = n \in \mathbb{N}_0$, $\tilde{\Omega}_n \in \mathbb{P}_n^{\mathbb{C}}$ and $\mu \in \mathbb{N}_0$, $\kappa \in \mathbb{N} \cup \{\infty\}$.*

(a) *Then the following two statements are equivalent*:

(i) *P_n and $\tilde{\Omega}_n$ satisfy the system*

$$\left.\begin{aligned} P_n(z)\,F(z) + \tilde{\Omega}_n(z) &= \dot{O}(z^{n+\mu}) \\ P_n^*(z)\,F(z) - \tilde{\Omega}_n^{(*)}(z) &= \dot{O}(z^{n+\kappa}) \end{aligned}\right\} \quad \text{as} \quad z \to 0. \tag{3.29}$$

(ii) $$\mathscr{L}(z^{-j} P_n) = 0, \qquad j \in [-(\kappa-1), ..., n+\mu-1],$$

and $\tilde{\Omega}_n$ is the polynomial of the second kind with respect to $\mathscr{L}$, i.e., $\tilde{\Omega}_n = \Omega_n$, where Ω_n is defined as in (1.35).

(b) *Suppose that the polynomial $P_n \in \mathbb{P}_n^{\mathbb{C}}$ of degree $n \in \mathbb{N}_0$ is orthogonal with respect to $\mathscr{L}$ and satisfies $\mathscr{L}(z^{-j} P_n) = 0$, $j \in [-(\kappa-1), ..., n+\mu-1]$, for some $\kappa, \mu \in \mathbb{N}$. Let $p \in \mathbb{N}_0$ denote the multiplicity of the zero of P_n at $z=0$. Then*

$$p \leqslant \mu - 1 \qquad \text{and} \qquad \kappa = \mu - p.$$

If we want to apply the general characterization-theorem 3.8(a) to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ we have first to show that

$$c_{-k}^{\lambda} = \mathscr{L}(x^k; \mathscr{A}, \mathscr{W}, \lambda) = \overline{\mathscr{L}(x^{-k}; \mathscr{A}, \mathscr{W}, \lambda)} = \overline{c_k^{\lambda}} \qquad \text{for all} \quad k \in \mathbb{N}_0. \tag{3.30}$$

The second property supposed in (3.16), i.e., $\sum_{k=0}^{\infty} c_k^{\lambda} z^k$ absolutely convergent on a nonempty subset of the unit circle containing $z=0$, has already been shown implicitly in the proof of Theorem 2.1.

Property (3.30) can be obtained in the following way: from (3.11) we have

$$F\left(\frac{1}{\bar{z}}; A, W, \lambda\right) = -\overline{F(z; A, W, \lambda)} \quad \text{for } |z| < \varrho := \min\{|z_j| : \lambda_j = -1\} > 0,$$

and therefore by (2.5)

$$-\mathscr{L}\left(\frac{\frac{1}{x}+\bar{z}}{\frac{1}{x}-\bar{z}};\mathscr{A},\mathscr{W},\lambda\right)=-\overline{\mathscr{L}\left(\frac{x+z}{x-z};\mathscr{A},\mathscr{W},\lambda\right)},$$

where $\mathscr{L}(\cdot;\mathscr{A},\mathscr{W},\lambda)$ acts on x.

By (3.22) the functional $\mathscr{L}(\cdot;\mathscr{A},\mathscr{W},\lambda)$ can be represented as a power series at $z=0$ and thus

$$\begin{aligned}&\mathscr{L}(1;\mathscr{A},\mathscr{W},\lambda)+2\sum_{k=1}^{\infty}\mathscr{L}(x^k;\mathscr{A},\mathscr{W},\lambda)\cdot\bar{z}^k\\&\quad=\overline{\mathscr{L}(1;\mathscr{A},\mathscr{W},\lambda)}+2\sum_{k=1}^{\infty}\overline{\mathscr{L}(x^{-k};\mathscr{A},\mathscr{W},\lambda)}\cdot\bar{z}^k.\end{aligned}$$

From the uniqueness of power series representations of analytic functions, relation (3.30) follows.

The following theorem gives, as mentioned above, another characterization of a polynomial to be orthogonal with respect to the linear functional $\mathscr{L}(\cdot;\mathscr{A},\mathscr{W},\lambda)$. This characterization, which in general is harder to verify, but which is on the other hand less restrictive than the conditions given in Theorem 2.2, is needed to prove Theorem 2.2.

THEOREM 3.9. *Let the polynomials* $R, V, W, A, B:=B(\cdot;A,W,\lambda)$ *and the linear functional* $\mathscr{L}(\cdot;\mathscr{A},\mathscr{W},\lambda)$, *where* $\lambda=(\lambda_1,...,\lambda_{m^*})\in\Delta_{m^*}$, *be given as in* (1.17), (2.2) *and* (3.8). *Further let* $P_n\in\mathbb{P}_n^{\mathbb{C}}$ *of degree* $n\geqslant\max\{\partial\mathscr{A}+l/2-w, 2v-l\}$ *and* $\mu\in\mathbb{N}_0$. *Then the following two properties are equivalent*:

(a) $\mathscr{L}(z^{-j}P_n;\mathscr{A},\mathscr{W},\lambda)=0$ *for* $j\in(0,...,n+\mu-1]$.

(b) *There exists a polynomial* $Q_{n+l-2v}\in\mathbb{P}_{n+l-2v}^{\mathbb{C}}$ *such that* P_n *and* Q_{n+l-2v} *fulfill the following conditions*:

$$\begin{aligned}&\text{(i)}\quad (VQ_{n+l-2v})^{(\nu)}(z_j)=\lambda_j(\sqrt{R}\,P_n)^{(\nu)}(z_j),\\&\qquad\quad \nu=0,...,m_j-1,\quad j=1,...,m^*\\&\text{(ii)}\quad V(z)\,Q_{n+l-2v}(z)-\sqrt{R(z)}\,P_n(z)\\&\qquad\quad =\dot{O}(z^{n-(\partial\mathscr{A}+l/2-w)+\mu}),\qquad \textit{as}\quad z\to 0\\&\text{(iii)}\quad V(z)\,Q^{(*)}_{n+l-2v}(z)-\sqrt{R(z)}\,P_n^*(z)\\&\qquad\quad =O(z^{n-(\partial\mathscr{A}+l/2-w)+1}),\qquad \textit{as}\quad z\to 0.\end{aligned}\tag{3.31}$$

Further the polynomial Q_{n+l-2v} *in* (3.31) *can be written explicitly in the form*

$$Q_{n+l-2v}(z)=\frac{i\Omega_n(z)\,A(z)-P_n(z)\,B(z)}{z^{a+l/2-w}},\tag{3.32}$$

where Ω_n *denotes the polynomial of the second kind with respect to* $\mathscr{L}(\cdot;\mathscr{A},\mathscr{W},\lambda)$.

Remark. From assumption (iii) in (3.31) it follows that $\partial Q_{n+l-2v}=n+l-2v$, thus we can write Q^{*}_{n+l-2v} instead of $Q^{(*)}_{n+l-2v}$.

Proof of Theorem 3.9. (a) $\Rightarrow$ (b) By Theorem 3.8(a), Theorem 2.1 and (3.11) we obtain

$$\left.\begin{aligned}
&z^{a+l/2-w}P_n(z)\frac{W(z)}{\sqrt{R(z)}}-(i\Omega_n(z)\,A(z)-P_n(z)\,B(z))\\
&\qquad=\dot{O}(z^{n+a-\partial\mathscr{A}+\mu})\\
&z^{a+l/2-w}P_n^{*}(z)\frac{W(z)}{\sqrt{R(z)}}-(-i\Omega_n^{(*)}(z)\,A(z)-P_n^{*}(z)\,B(z))\\
&\qquad=O(z^{n+a-\partial\mathscr{A}+1})
\end{aligned}\right\}\quad\text{as}\quad z\to 0.\tag{3.33}$$

From this system it follows that (note that $n+a-\partial\mathscr{A}\geqslant a+l/2-w$)

$$\frac{i\Omega_n A-P_n B}{z^{a+l/2-w}}\in\mathbb{P}^{\mathbb{C}}\qquad\text{and}\qquad-\frac{i\Omega_n^{(*)}A+P_n^{*}B}{z^{a+l/2-w}}\in\mathbb{P}^{\mathbb{C}}.$$

Together with

$$\left(\frac{i\Omega_n A-P_n B}{z^{a+l/2-w}}\right)^{(*)}_{n+a-l/2+w}=-i\Omega_n^{(*)}A-P_n^{*}B$$

one gets

$$\partial\left(\frac{i\Omega_n A-P_n B}{z^{a+l/2-w}}\right)\leqslant n+a-\frac{l}{2}+w-\left(a+\frac{l}{2}-w\right)=n+l-2v.\tag{3.34}$$

Thus Q_{n+l-2v} from (3.32) is in fact a polynomial of degree $\leqslant n+l-2v$. Now from (3.33) and (3.34) the second and the third property in (3.31) follow and by

$$\begin{aligned}
iV(z)\,\Omega_n(z)\,A(z)&=V(z)(z^{a+l/2-w}Q_{n+l-2v}(z)+P_n(z)\,B(z))\\
&=z^{a+l/2-w}(V(z)\,Q_{n+l-2v}(z)-\lambda_j\sqrt{R(z)}\,P_n(z))\\
&\quad+P_n(z)(V(z)\,B(z)+\lambda_j z^{a+l/2-w}\sqrt{R(z)})
\end{aligned}\tag{3.35}$$

and by the interpolation property (3.8) we get (3.31)(i).

(b) $\Rightarrow$ (a) As in the proof of Theorem 2.1 let us denote $z_0 := 0$, $m_0 := a - \partial\mathscr{A}$ and $\lambda_0 := 1$. By the assumptions (i) and (ii) in (3.31), by the second identity in (3.35), by the interpolation property in (3.8) and by the fact that V and A have no zero in common, the polynomial $z^{a+l/2-w} Q_{n+l-2v}(z) + P_n(z)\, B(z)$ has a root at z_j of multiplicity m_j, $j = 0, ..., m^*$, thus

$$\frac{z^{a+l/2-w} Q_{n+l-2v}(z) + P_n(z)\, B(z)}{iA(z)} \in \mathbb{P}^{\mathbb{C}}_{n+2a-(2a)} = \mathbb{P}^{\mathbb{C}}_n. \tag{3.36}$$

Now we get from Theorem 2.1, (3.11), (1.4) and Property (ii) in (3.31)

$$\begin{aligned} &P_n(z)\, F(z; A, W, \lambda) + \left(\frac{z^{a+l/2-w} Q_{n+l-2v}(z) + P_n(z)\, B(z)}{iA(z)} \right) \\ &\quad = \dot{O}(z^{n+\mu}), \qquad \text{as} \quad z \to 0. \end{aligned} \tag{3.37}$$

From $R = R^*$, $V = V^*$, $A = A^{(*)}_{2a}$, and assumption (3.31)(i) it follows that

$$(V Q^*_{n+l-2v})^{(v)}\,(z_j) = \lambda_j (\sqrt{R}\, P^*_n)^{(v)}\,(z_j), \qquad v = 0, ..., m_j - 1, \quad j = 1, ..., m^*.$$

With the help of these equations and assumption (3.31)(ii) we obtain in a similar way as in (3.37)

$$\begin{aligned} &P^*_n(z)\, F(z; A, W, \lambda) - \left(\frac{z^{a+l/2-w} Q_{n+l-2v}(z) + P_n(z)\, B(z)}{iA(z)} \right)^{(*)}_n \\ &\quad = O(z^{n+1}), \qquad \text{as} \quad z \to 0. \end{aligned} \tag{3.38}$$

Now the orthogonality property in part (a) follows from (3.36)–(3.38), Theorem 2.1 and Theorem 3.8(a). ∎

4. Proofs of Theorem 2.2, Corollary 2.4, and Theorem 2.5

Proof of Theorem 2.2. For technical reasons we show the implications (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (b), and (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c) By Theorem 3.8(a), Theorem 2.1 and (3.11) the orthogonality property from part (a) of the theorem is equivalent to the system (note that κ from Theorem 3.8(a) is $< \infty$)

$$\left.\begin{aligned} P_n(z)\, F(z; A, W, \lambda) + \Omega_n(z) &= \dot{O}(z^{n+\mu}) \\ P^*_n(z)\, F(z; A, W, \lambda) - \Omega^{(*)}_n(z) &= O(z^{n+1}) \not\equiv 0 \end{aligned}\right\} \quad \text{as} \quad z \to 0. \tag{4.1}$$

From the first equation in (4.1) we get by (3.11)

$$i\Omega_n(z)\,A(z) - P_n(z)\,B(z) = z^{a+l/2-w}\,\frac{W(z)}{\sqrt{R(z)}}\,P_n(z) + \dot{O}(z^{n+\mu+a-\partial\mathscr{A}}). \quad (4.2)$$

We first consider

Case 1: $p < n - (\partial\mathscr{A} + l/2 - w) + \mu$. Squaring the equation in (4.2) yields

$$W(z)\,P_n^2(z) - V(z)\left[\frac{i\Omega_n(z)\,A(z) - P_n(z)\,B(z)}{z^{a+l/2-w}}\right]^2$$
$$= \dot{O}(z^{n+p-(\partial\mathscr{A}+l/2-w)+\mu}), \quad (4.3)$$

where we have shown in Theorem 3.9 that

$$Q_{n+l-2v}(z) := \frac{i\Omega_n(z)\,A(z) - P_n(z)\,B(z)}{z^{a+l/2-w}} \in \mathbb{P}^{\mathbb{C}}_{n+l-2v}. \quad (4.4)$$

Squaring out the bracket-term in (4.3), multiplying the whole equation by $z^{2(a+l/2-w)}$ and using the fact that because of (3.8) the polynomial $VB^2 - z^{2(a+l/2-w)}W$ vanishes at the zeros of A, yields that A divides the left-hand side of (4.3), i.e., we can write

$$W(z)\,P_n^2(z) - V(z)\,Q^2_{n+l-2v}(z) = z^{n+p-(a+l/2-w)+\mu}A(z)\,g_{(n)}(z), \quad (4.5)$$

where $g_{(n)} \in \mathbb{P}^{\mathbb{C}}_{n-p+w+l/2-a-\mu}$ and $g_{(n)}(0) \neq 0$. By taking the modified reciprocal polynomials with respect to $\mathbb{P}^{\mathbb{C}}_{2n+2w}$ at both sides of the Equation (4.5) we get

$$W(z)\,P_n^{*2}(z) - V(z)\,Q^{(*)\,2}_{n+l-2v}(z) = A(z)\,g^{(*)}_{(n),n-p+w+l/2-a-\mu}(z). \quad (4.6)$$

An analog method used in (4.3) and (4.5) applied to the second equation in (4.1) leads to

$$W(z)\,P_n^{*2}(z) - V(z)\,Q^{(*)2}_{n+l-2v}(z) = z^{n-(a+l/2-w)+1}A(z)\,h_{(n)}(z), \quad (4.7)$$

where $h_{(n)} \in \mathbb{P}^{\mathbb{C}}_{n+w+l/2-a-1}$ and $h_{(n)} \not\equiv 0$ (note that the right-hand side of the second equation in (4.1) does not vanish identically). From the above two equations (4.6) and (4.7) it follows that

$$g^{(*)}_{(n),n-p+w+l/2-a-\mu}(z) = z^{n-(a+l/2-w)+1}h_{(n)}(z);$$

thus

$$\partial g_{(n)} \leqslant n-p+w+\frac{l}{2}-a-\mu-\left(n-\left(a+\frac{l}{2}-w\right)+1\right)=l-p-\mu-1.$$

Recall now that the left hand side of the second equation in (4.1) is of the form $\dot{O}(z^{n+\kappa})$ with $\kappa \in \mathbb{N}$. Then we obtain with the help of Theorem 3.8(b) that for $\mu>0$ one even has $\partial g_{(n)} \leqslant l-p-\mu-\kappa=l-2\mu$. Further, again by Theorem 3.8(b), we have $\mu \geqslant p+1$. Thus by (4.5) and (4.4) relation (2.10) is proved. The assertion (2.11) can be seen from (4.2).

Next we consider

Case 2: $p \geqslant n-(\partial \mathscr{A}+l/2-w)+\mu$. Since by Theorem 3.8(b) $\mu-p \geqslant 1$ for $\mu>0$, case 2 is only possible for $\mu=0$. Again, in a similar way as in case 1 again we get equation (4.7) and instead of (4.6) the equation

$$W(z)\, P_n^{*2}(z)-V(z)\, Q_{n+l-2v}^{(*)2}(z)=A(z)\, g_{(n), l-(a-\partial \mathscr{A})}^{(*)}(z),$$

where $g_{(n)} \in \mathbb{P}_{l-(a-\partial \mathscr{A})}^{\mathbb{C}}$, i.e.,

$$g_{(n), l-(a-\partial \mathscr{A})}^{(*)}(z)=z^{n-(a+l/2-w)+1} h_{(n)}(z).$$

From the last identity it follows that

$$\partial g_{(n)} \leqslant l-(a-\partial \mathscr{A})-\left(n-\left(a+\frac{l}{2}-w\right)+1\right)=\partial \mathscr{A}+\frac{3l}{2}-n-w-1.$$

Since $g_{(n)} \not\equiv 0$ we have $\partial \mathscr{A}+3l/2-n-w-1 \geqslant 0$, i.e., $n \leqslant \partial \mathscr{A}+3l/2-w-1$. Thus by $v+w=l$ case 2 can only occur for $n<\partial \mathscr{A}+l/2+v$, which is not contained in the assumptions of this theorem.

(c) $\Rightarrow$ (b) From the quadratic Equation (2.10) we obtain that Q_{n+l-2v}, given as in (4.4), is a polynomial of degree $n+l-2v$ satisfying (2.7) and by (3.35) and (3.8) Q_{n+l-2v} fulfills the interpolation property (2.8). Further from (2.11) the first condition in (2.9) follows. By the definition of Ω_n in (1.35) one has in any case

$$P_n^*(z)\, F(z; A, W, \lambda)-\Omega_n^{(*)}(z)=O(z^n), \tag{4.8}$$

where $F(z; A, W, \lambda)$ is given as in (3.11), and thus for $n>\partial \mathscr{A}+l/2-w$ the second condition in (2.9) is also proved. For the remaining case, i.e., $a=\partial \mathscr{A}$ and $n=a+l/2-w$, let us denote $P_n(z)=\alpha_n z^n+\cdots+\alpha_0$, $\Omega_n(z)=\beta_n z^n+\cdots+\beta_0$ and $F(z; A, W, \lambda)=c_0^\lambda+2\sum_{j=1}^{\infty} c_j^\lambda z^j$. From (2.11) we see (note $\partial \mathscr{A}+l/2-w+1>0$)

$$\beta_0=-\alpha_0 c_0^\lambda \tag{4.9}$$

and on the other hand again from (1.35)

$$\beta_0 = \alpha_1 \overline{c_1^\lambda} + \alpha_2 \overline{c_2^\lambda} + \cdots + \alpha_n \overline{c_n^\lambda}. \tag{4.10}$$

Comparing (4.9) and (4.10) we have $\alpha_0 c_0^\lambda + \alpha_1 \overline{c_1^\lambda} + \cdots + \alpha_n \overline{c_n^\lambda} = 0$ and the coefficient of z^n on the left hand side in (4.8) vanishes, i.e.,

$$\overline{\alpha_0} c_0^\lambda + 2\overline{\alpha_1} c_1^\lambda + \cdots + 2\overline{\alpha_n} c_n^\lambda - \overline{\beta_0} = 0. \tag{4.11}$$

Thus the left hand side in (4.8) is of the order $O(z^{n+1})$, note $n = a + l/2 - w$, and again the second assertion in (2.9) follows.

(b) $\Rightarrow$ (a) We show that the system (3.31) is fulfilled which, by Theorem 3.9, implies part (a). First let us note that because of $g_{(n)} \not\equiv 0$ the left hand side in (3.31)(iii) cannot vanish identically and thus by Theorem 3.8(b) there exists a $\tau \in \mathbb{N}$ such that $\mathscr{L}(z^\tau P_n; \mathscr{A}, \mathscr{W}, \lambda) \neq 0$.

Concerning (3.31)(i) let us first consider the case $P_n(z_j) \neq 0$ for a fixed z_j. From the quadratic Equation (2.7) we have for $v = 0, ..., m_j - 1$

$$\begin{aligned} 0 &= ((VQ_{n+l-2v} - \lambda_j \sqrt{R}\, P_n)(VQ_{n+l-2v} + \lambda_j \sqrt{R}\, P_n))^{(v)}(z_j) \\ &= \sum_{k=0}^{v} \binom{v}{k} (VQ_{n+l-2v} - \lambda_j \sqrt{R}\, P_n)^{(k)}(z_j) \\ &\quad \times (VQ_{n+l-2v} + \lambda_j \sqrt{R}\, P_n)^{(v-k)}(z_j). \end{aligned} \tag{4.12}$$

Since $V(z_j)\, Q_{n+l-2v}(z_j) - \lambda_j \sqrt{R(z_j)}\, P_n(z_j) = 0$ and $\lambda_j \sqrt{R(z_j)}\, P_n(z_j) \neq 0$ (note that A has no zero in Γ_{E_l}) one has

$$V(z_j)\, Q_{n+l-2v}(z_j) + \lambda_j \sqrt{R(z_j)}\, P_n(z_j) \neq 0,$$

and (3.31)(i) follows from (4.12). If $P_n(z_j) = 0$ then (3.31)(i) follows immediately by (2.8). Considering condition (3.31)(ii) we see by the first property in (2.9) that the polynomial Q_{n+l-2v} must have a zero of exact order p at $z = 0$. Therefore we can write for $|z| < 1$ (recall that $\sqrt{R}$ is analytic in $|z| < 1$):

$$\begin{aligned} V(z)\, Q_{n+l-2v}(z) &=: a_p z^p + a_{p+1} z^{p+1} + \cdots \\ \sqrt{R(z)}\, P_n(z) &=: b_p z^p + b_{p+1} z^{p+1} + \cdots \end{aligned} \qquad \text{and} \qquad a_p = b_p \neq 0.$$

Substituting these representations in Equation (2.7), which reads after multiplying with V as

$$R(z)\, P_n^2(z) - V^2(z)\, Q_{n+l-2v}^2(z) = \dot{O}(z^{n+p-(\partial\mathscr{A}+l/2-w)+\mu}),$$

and comparing coefficients, leads to (3.31)(ii).

Finally, multiplying again the quadratic Equation (2.7) with V and calculating the modified reciprocal polynomials with respect to $\mathbb{P}^{\mathbb{C}}_{2n+2l}$, yields

$$R(z)\, P_n^{*2}(z) - V^2(z)\, Q_{n+l-2v}^{*2}(z) = \begin{cases} z^{n-p-(a+l/2-w)+\mu} V(z)\, A(z)\, g^{(*)}_{(n), l-2\mu}(z), & \mu > 0 \\ z^{n-(a+l/2-w)+1} V(z)\, A(z)\, g^{(*)}_{(n), l-p-1}(z), & \mu = 0. \end{cases}$$

Comparing coefficients and using the fact that $V(0)\, Q_{n+l-2v}^{*}(0) = \sqrt{R(0)}\, P_n^*(0) \neq 0$, we get from the above identity

$$\sqrt{R(z)}\, P_n^*(z) - V(z)\, Q_{n+l-2v}^*(z) = \begin{cases} O(z^{n-(\partial \mathscr{A}+l/2-w)+(\mu-p)}), & \mu > 0 \\ O(z^{n-(\partial \mathscr{A}+l/2-w)+1}), & \mu = 0. \end{cases}$$

This is (3.31)(iii); recall $\mu - p \geqslant 1$ by Theorem 3.8(b). ▮

Proof of Corollary 2.4. Since $f(\varphi; \mathscr{A}, \mathscr{V})$ is integrable and $v + w = l$ the trigonometric polynomial $\mathscr{V}$ fulfills both restriction in Assumption 1.1. Comparing (2.1), let

$$a(\mathscr{V}) := \max \left\{ \partial \mathscr{A}, v - \frac{l}{2} \right\}$$

and define, note (2.2),

$$A(z; \mathscr{V}) := z^{a(\mathscr{V}) - \partial \mathscr{A}} \tilde{A} \in \mathbb{P}^{\mathbb{C}}_{2a(\mathscr{V})},$$

where $\tilde{A}$ is given by $\tilde{A}(e^{i\varphi}) := e^{i\,\partial \mathscr{A}\,\varphi} \mathscr{A}(\varphi)$. By (1.4) the system (3.31) can be rewritten as (note $n + l - 2v \geqslant \max\{\partial \mathscr{A} + l/2 - v, 2w - l\}$)

$$(WP_n)^{(\nu)}(z_j) = \lambda_j (\sqrt{R}\, Q_{n+l-2v})^{(\nu)}(z_j),$$
$$\nu = 0, ..., m_j - 1, \quad j = 1, ..., m^*$$
$$W(z)\, P_n(z) - \sqrt{R(z)}\, Q_{n+l-2v}(z) = \dot{O}(z^{(n+l-2v)-(\partial \mathscr{A}+l/2-v)+\mu}), \quad \text{as} \quad z \to 0$$
$$W(z)\, P_n^*(z) - \sqrt{R(z)}\, Q_{n+l-2v}^*(z) = O(z^{(n+l-2v)-(\partial \mathscr{A}+l/2-v)+1}), \quad \text{as} \quad z \to 0.$$

Since $\partial Q_{n+l-2v} = n + l - 2v$, see the remark after Theorem 3.9, the assertion follows from Theorem 3.9. ▮

Proof of Theorem 2.5. The representations (2.15) and (2.16) follow from (recall (2.13) resp. (3.32))

$$z^{a+l/2-w} Q_{n+l-2v}(z) = i\Omega_n(z)\, A(z) - P_n(z)\, B(z; A, W, \lambda)$$

and from the representations of Ω_n in (1.35) (note $n \geqslant 1$) and $B(\cdot; A, W, \lambda)$ in Corollary 3.6 by straightforward calculation. Part (b) can be seen by changing the roles of P_n, Q_{n+l-2v} and W, V, respectively. ∎

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